Minimal Saturated Subgraphs of the Hypercube

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Abstract
Within the hypercube $Q_n$, we investigate bounds on the saturation number of a forbidden graph $G$, defined as the minimum number of edges in a subgraph $H$ of $Q_n$ that is both $G$-free and has the property that the addition of any $e \in E(Q_n) \cap E(H)$ creates $G$. For all graphs $G$, we find a lower bound based on the minimum degree of non-leaves. For upper bounds, we first examine general graphs and derive conditions that, if satisfied, allow us to bound the saturation number. We also study specific cases, finding improved bounds for paths, stars, and most caterpillars. In all of these cases, we find bounds that are $O(2^n)$, an interesting fact that we conjecture to hold for all trees $T$.

Summary
We study properties of the hypercube, an extension of the cube to any number of dimensions. The hypercube is especially interesting because a direct connection can be made between it and parallel computer networks. Our problem, in a parallel network, is equivalent to finding the maximum number of links that must fail before some desired configuration of processors and links no longer exists. This value is directly associated with the stability of the parallel architecture. We provide improved bounds on this maximum for many desired structures, along with devising novel methods for finding both lower and upper bounds.