Concatenations of Two Incidence Matrices, Spanning Trees in Planar Graphs: Two Related Problems

Alan Bu

Under the direction of

Yuchong Pan
Department of Mathematics
Massachusetts Institute of Technology

Research Science Institute
July 29, 2023

Abstract

We call a matrix an *incidence submatrix* if each row has at most one 1 and at most one -1. In this paper, we establish an interesting connection between the concatenation of two incidence submatrices and spanning trees in a planar multigraph, namely that the maximum determinant of such an $m \times m$ concatenation is at least the maximum number of spanning trees in a planar multigraph with m edges. In addition, we present some evidence supporting our conjecture that these two quantities might indeed be equal. This includes several algebraic results on nonplanar graphs. Finally, we give several nontrivial asymptotic lower and upper bounds on these two quantities, showing that they both grow exponentially in m.

Summary

Graphs are universal structures in real life abstracting pairwise relations between objects, such as road systems, electronic circuits, and railway maps. Starting with programming experiments, we observed and proved an interesting, undiscovered connection between two fields of mathematics, linear algebra and graph theory. Concretely, if we concatenate two matrices in which each row has at most one 1 and at most one -1, then the maximum possible determinant of a matrix generated this way is at least the maximum number of certain structures in a graph that can be drawn on a plane without edge crossings. We also present some evidence supporting our conjecture that these two quantities might indeed be equal. Finally, we demonstrate that these two quantities both grow exponentially in m.

1 Introduction

Total unimodularity is an extremely powerful tool for proving integrality of a polyhedron in combinatorial optimization. We say that a matrix is *totally unimodular* if the determinant of every square submatrix is in $\{-1, 0, 1\}$. A common theme in combinatorial optimization is to embed combinatorial objects as their characteristic vectors in the Euclidean space and to relax combinatorial problems into linear programs, which can be solved in polynomial time. Hence, studying properties of the resulting coefficient matrix, such as total unimodularity, sometimes sheds light on properties of the corresponding polyhedron and the original combinatorial problem. In particular, given a polyhedron $P = \{x : Ax \leq b\}$, if the matrix A is totally unimodular and vector b has integer components, then all of the extreme points of P have integer components.

Given a directed multigraph $D = (V, A)$ with $V = \{v_1, \ldots, v_n\}$ and $A = \{a_1, \ldots, a_m\}$, we define its *incidence matrix* M_D to be an $m \times n$ matrix given by $(a_{ij})_{i,j}$ where $a_{ij} = 1$ or -1 when vertex v_j is the head or tail of arc a_i, respectively, and 0 otherwise. We say that a matrix is an *incidence matrix* if it is the incidence matrix of some directed graph. It is well known that incidence matrices are *totally unimodular*. However, given two incidence matrices M_1 and M_2, it is not immediately clear whether their concatenation $M = [M_1|M_2]$ is totally unimodular. This turns out to not be true. Indeed, Goemans and Pan [2] gave a 3×3 submatrix of such a concatenation with determinant -3, and constructed a counterexample from this matrix to the conjecture proposed by A. Abdi, G. Cornuéjols, and G. Zambelli in 2023 that the intersection of two crossing submodular flow systems is box-half-integral.[1][2]

Following their work, we investigate how large or small can such a determinant be. Concretely, we study the following problem:

Problem 1.1. *What is the largest (respectively, smallest) determinant of a square matrix formed by concatenating two matrices which are submatrices of incidence matrices?*

1.1 Organization of the Paper

In Section 2, we introduce definitions, notations and known theorems to be used in subsequent sections.

In Section 3, we show that the maximum determinant $\max\det_m$ of an $m \times m$ matrix formed by concatenating two matrices which are submatrices of incidence matrices is at least the largest number $\max \mathrm{sp}_m^{\mathrm{planar}}$ of spanning trees in a planar multigraph with m edges. Furthermore, we show that the construction used in proving the above inequality is the best possible in the sense that the maximum possible determinant of a square concatenation with

[1]A polyhedron $P \subseteq \mathbb{R}^n$ is said to be *box-half-integral* if the extreme points of the intersection of P and integer box constraints $\ell \leq x \leq u$ for some $\ell, u \in \mathbb{Z}^n$ have only half-integral components.

[2]The conjecture was proposed by A. Abdi, G. Cornuéjols, and G. Zambelli in the talk titled *Arc connectivity and submodular flows in digraphs* at the "Combinatorics and Optimization" workshop at ICERM in 2023. The slides of the talk are available at `https://app.icerm.brown.edu/assets/403/4951/4951_3700_Abdi_032820231400_Slides.pdf`, and the video of the talk is available at `https://icerm.brown.edu/video_archive/?play=3092`.

the left incidence submatrix being the truncated incidence submatrix of a planar multigraph G is at most the number of spanning trees in G. We conjecture that $\max\det_m = \max \mathrm{sp}_m^{\text{planar}}$.

In Section 4, we prove several theorems as evidence supporting this conjecture. We show that the maximum possible determinant of a square concatenation with the left incidence submatrix being the truncated incidence submatrix of a nonplanar multigraph G is at most the number of spanning trees in G minus 18. We also show that when a nonplanar graph lies in certain special classes, namely subdivisions of K_5 or $K_{3,3}$, it always underperforms some planar graph.

In Section 5, we prove several nontrivial asymptotic bounds on these two quantities, showing that they grow exponentially as a function of m.

2 Preliminaries

We say that a vector is *integral* if all of its components are integers. Given a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, its *fractionality* is defined to be $\min\{x_i : i \in [n], x_i > 0\}$. For a given matrix M, we let $M_{i,j}$ denote the minor of M obtained by removing the ith row and jth column. For an $m \times n$ matrix M and subsets $S_1 \subseteq [m], S_2 \subseteq [n]$, we denote the submatrix of M whose rows are restricted to S_1 and whose columns are restricted to S_2 as $M[S_1, S_2]$. We let $\binom{[m]}{n}$ denote the set of n-element subsets of $[m]$. Throughout this paper, we forbid multigraphs from having loops.

2.1 Incidence Matrices

Given a directed multigraph $D = (V, A)$ with $V = \{v_1, \ldots, v_n\}$ and $A = \{a_1, \ldots, a_m\}$, we define its *incidence matrix* M_D to be an $m \times n$ matrix given by $(a_{ij})_{i,j}$ where $a_{ij} = 1$ or -1 if vertex v_j is the head or tail of arc a_i, respectively, and 0 otherwise. We say that a matrix is an *incidence matrix* if it is the incidence matrix of some directed graph. Furthermore, we define its *truncated incidence matrix* M'_D to be M_D with its first column removed. Note that any matrix M such that every row of M has at most one 1 and at most one -1 can be written as the truncated incidence matrix of a unique directed graph along with some rows of all 0s. To obtain this graph, we simply add a column to M with entries in $\{-1, 0, 1\}$ such that the sum of the entries in each row is equal to 0. Thus each row is either empty or has exactly one 1 and one -1.

For an $m \times n$ matrix A and an $m \times k$ matrix B, we define their concatenation $M = [A|B]$ to be a $m \times (n + k)$ matrix such that for each $i \in [m]$, the ith row of M is the concatenation of the ith row of A with the ith row of B.

For every positive integer m, we define $\max\det_m$ to be the maximum value of $\det M$ over all $m \times m$ matrices M which are the concatenation of two truncated incidence matrices. Note that the minimum value of $\det M$ is simply $-1 \cdot \max\det_m$ since we can flip the signs in an arbitrary row of M to flip the sign of the determinant. It follows that $\max\det_m \geq 0$.

Given a connected multigraph G, we use $\max\det G$ to denote the maximum determinant of $[M'_D|N]$ over all orientations D of G and over all truncated incidence matrices N such that $[M'_D|N]$ is a square matrix. If G is not connected, we define $\max\det G = 0$. Note that the specific orientation of G chosen is irrelevant as given any two orientations D_1, D_2 of G, we can multiply some rows of M_{D_1} by ± 1 so that they match with the corresponding rows of M_{D_2}. These operations keep N as a truncated incidence matrix and do not change the absolute value of the determinant, so we can fix an orientation D of G and define $\max\det G$ to be the largest absolute value of the determinant of $[M'_D|N]$ for a fixed orientation D and over all incidence submatrices N. The chosen truncated column does not matter since the sum of the columns of M_D is the zero vector (each nonempty row has exactly one 1 and one -1), so any M'_D can be obtained through column sum operations on another choice of M'_D.

2.2 Spanning Trees and Kirchhoff's Matrix-Tree Theorem

Let $G = (V, E)$ be a multigraph. We denote the number of spanning trees of G as $\mathrm{sp}(G)$. An edge contraction along an edge of G involves deleting all edges between the endpoints of the edge and merging the two endpoints in G. For any edge $e = (u, v) \in E$, we define G^e_{cut} and G^e_{merge} to be the multigraphs obtained by removing and contracting edge e, respectively.

Let $n = |V|$. Without loss of generality, we assume that $V = [n]$. We define the *adjacency matrix* of G to be an $n \times n$ matrix whose (i, j)-entry is the number of edges between vertex i and vertex j for all $i, j \in [n]$. We define the *Laplacian matrix* of G to be an $n \times n$ matrix $L = D - A$, where D is the diagonal matrix whose (i, i)-entry is the degree of vertex i for $i \in [n]$, and A is the adjacency matrix of G.

The following celebrated theorem by Kirchhoff [3] allows one to count the number of spanning trees in any graph G algebraically.

Theorem 2.1 (Kirchhoff's matrix-tree theorem, 1847, [3])**.** *Let L be the Laplacian matrix of a multigraph G. Then* $\det L_{1,1} = \mathrm{sp}(G)$*.*

In fact, the number of spanning trees in a graph G has a deletion-contraction relation which is used in the proof of Kirchhoff's matrix-tree theorem and is useful to us.

Theorem 2.2 (Lewin, 1982, [4])**.** *For any multigraph $G = (V, E)$ and edge $e \in E$, we have*

$$\mathrm{sp}(G) = \mathrm{sp}(G^e_{\text{merge}}) + \mathrm{sp}(G^e_{\text{cut}}).$$

2.3 Planar Multigraphs

A multigraph G is said to be *planar* if it can be embedded into the plane in a way so that any two edges of G can only intersect at a vertex.

For a directed planar multigraph D, we can construct its directed planar dual D^* by taking the faces of a planar embedding of the underlying undirected graph of D to be the vertices of D^*. We then rotate each edge of D by 90° counterclockwise to obtain a directed graph on D^* with a bijection between the edges of the two planar multigraphs. For ease of notation, let the ith rows of M_{D^*}, M'_{D^*} correspond to the same edge as the ith rows of M_D, M'_D respectively when D^* is constructed as the dual of D. In particular, we force rows in M_{D^*}, M'_{D^*} which correspond to loops in D^* to only have zero entries, effectively removing them from consideration. We also define the undirected planar dual G^* of G as the underlying undirected graph of D^* for any orientation D of G.

In particular, Euler's formula implies that the planar dual G^* of some graph $G = (V, E)$ has exactly $|E| + 2 - |V|$ vertices. The following theorem by Tutte [7] relates the numbers of spanning trees in a planar dual pair.

Theorem 2.3 (Tutte, 1984, [7])**.** *Let G be a connected planar multigraph and G^* its planar dual. We have that* $\mathrm{sp}(G) = \mathrm{sp}(G^*)$*.*

If the graph obtained from contracting or deleting any edge of G (i.e., any minor of G) is nonplanar, then G is also nonplanar. Wagner's theorem allows us to characterize all nonplanar graphs using this property, and is useful for analyzing general nonplanar graphs.

Theorem 2.4 (Wagner's theorem, 1937, [8])**.** *Given any nonplanar graph G, there exists a sequence of edge-deletions and edge-contractions to obtain either a $K_{3,3}$ or a K_5.*

For any multigraph $G = (V, E)$, we define the *cycle space* $\mathcal{C}(G)$ of G to be the space of subgraphs of G such that every vertex of G has even degree in the subgraph. This space is a vector space in $\mathbb{F}_2^{|E|}$. We define a 2*-basis* of $\mathcal{C}(G)$ to be a basis of $\mathcal{C}(G)$ such that each edge appears in at most two elements in the basis. Mac Lane [5] gave the following alternative characterization of planar graphs:

Theorem 2.5 (Mac Lane's planarity criterion, 1936, [5])**.** *A multigraph G is planar if and only if its cycle space has a 2-basis.*

3 Proving $\max \det_m \geq \max \mathrm{sp}_m^{\mathrm{planar}}$

Using a computer program, we computed $\max \det_m, \max \mathrm{sp}_m^{\mathrm{planar}}$ and $\max \mathrm{sp}_m^{\mathrm{general}}$ for $m \in [10]$, and the results are given in Table 1. It is surprising that the values of $\max \det_m$ and $\max \mathrm{sp}_m^{\mathrm{planar}}$ match up for $m \in [10]$. This result suggests that there might be some connection between the number of spanning trees on planar multigraphs with the determinant of the concatenation of incidence submatrices. In this section, we prove that $\max \det_m$ is always lower bounded by $\max \mathrm{sp}_m^{\mathrm{planar}}$. First, we prove the following theorem:

Theorem 3.1. *Let G be a connected planar multigraph. Let D be an orientation of G. Then*

$$\det \left[\begin{array}{c|c} M'_D & M'_{D^*} \end{array} \right] = \pm \, \mathrm{sp}(G).$$

Proof. It is clear that G has at least one spanning tree. Suppose G has m edges and n vertices. Notice that by Euler's formula, M'_{D^*} is an $m \times (m - n + 1)$ matrix, so $[M'_D | M'_{D^*}]$ is a square matrix. Let c_i instead denote the ith column vector of M_D and let v_i be the vertex corresponding to c_i. Now, consider the product $L_G = M_D^T M_D$. Let the element in the ith row and jth column be denoted l_{ij}. Then $l_{ij} = c_i \cdot c_j$ for all $1 \leq i, j \leq n$. Let c_{ij} denote the jth element of c_i. Then

$$l_{ii} = \sum_{j=1}^{m} c_{ij}^2 = \sum_{c_{ij} \neq 0} 1 = \deg v_i.$$

In any row, if $c_{ik}, c_{jk} \neq 0$ and $i \neq j$ then $c_{ik} \cdot c_{jk} = -1$ since c_{ik}, c_{jk} cannot both be 1 or both be -1. So for any $i \neq j$ we have

$$l_{ij} = \sum_{k=1}^{m} c_{ik} \cdot c_{jk} = \sum_{\substack{k \in [m] \\ c_{ik}, c_{jk} \neq 0}} c_{ik} \cdot c_{jk} = \sum_{\substack{k \in [m] \\ c_{ik}, c_{jk} \neq 0}} -1,$$

which is exactly the negative of the number of edges between v_i and v_j in G. Hence L_G is exactly the Laplacian matrix of G.

Similarly, the Laplacian matrix L_{G^*} of G^* is equal to $M_{D^*}^T M_{D^*}$. Suppose the truncated column of M_D is exactly its first column. Then the Kirchhoff matrix-tree theorem [1] implies that $\det M_D'^T M_D' = \det L_{1,1} = \mathrm{sp}(G) \neq 0$, so it follows that the column vectors of M_D' are linearly independent. From Theorem 2.3, we know that the number of spanning trees of G and G^* are equal, so by symmetry it follows that $\det M_{D^*}'^T M_{D^*}' = \mathrm{sp}(G^*) = \mathrm{sp}(G) \neq 0$ so the column vectors of M_{D^*}' are also linearly independent.

Now, we claim that M_D' and M_{D^*}' are orthogonal complements of one another. It suffices to show that for any column vector c_1 of M_D' and column vector c_2 of M_{D^*}', we have that $c_1 \cdot c_2 = 0$. Let $v \in V$ be the vertex of G corresponding to c_1 and f be the face of G which is dual to the vertex in G^* corresponding to c_2. Multiplying both the ith row of M_D' and the ith row of M_{D^*}' by -1 does not change the fact that the two directed graphs are duals of one another.

Without loss of generality, take an orientation D such that all edges incident to v have their head at v. If there is no edge e such that both v, f are incident to e, then $c_1 \cdot c_2 = 0$. Otherwise, there exists an edge e such that v lies on e and e lies on the face f. It follows that the vertex v lies on face f. The edges touching face f form a cycle on G, so v, f share exactly two edges. It follows from the definition of the directed planar dual that one of them has its tail at f and the other has its head at f. Hence $c_1 \cdot c_2 = 1 \cdot 1 + 1 \cdot -1 = 0$ as desired.

Now, consider the following equality:

$$\begin{aligned}
\det \left[\, M_D' \mid M_{D^*}' \,\right]^2 &= \det \left[\, M_D' \mid M_{D^*}' \,\right]^T \left[\, M_D' \mid M_{D^*}' \,\right] \\
&= \det \begin{bmatrix} M_D'^T \\ M_{D^*}'^T \end{bmatrix} \left[\, M_D' \mid M_{D^*}' \,\right] \\
&= \det \begin{bmatrix} M_D'^T M_D' & 0 \\ 0 & M_{D^*}'^T M_{D^*}' \end{bmatrix} \\
&= \det M_D'^T M_D' \cdot \det M_{D^*}'^T M_{D^*}' \\
&= \mathrm{sp}(G)^2.
\end{aligned}$$

It follows that $\det \left[\, M_D' \mid M_{D^*}' \,\right] = \pm\, \mathrm{sp}(G)$ as desired. $\square$

The above proof requires the use of a planar dual, which is ill-defined for nonplanar graphs and is thus hard to use to understand the behavior of nonplanar graphs. We also present a proof through the cycle space of G in Appendix B, which exists regardless of the planarity of G, and is potentially more generalizable.

Corollary 3.2. *For any positive integer m, we have that* $\max \det_m \geq \max \mathrm{sp}_m^{\mathrm{planar}}$.

Furthermore, we can ask whether it is possible to construct a square matrix M formed by concatenating the truncated incidence matrix of planar multigraph G with another incidence submatrix to obtain a larger determinant. It turns out that Theorem 3.1 gives the best possible bound:

Theorem 3.3. *Given a multigraph G, we have that* $\max \det G \leq \mathrm{sp}(G)$.

Proof. Suppose G has m edges and n vertices. Consider any matrix $M = [M'_D|N]$ such that D is an orientation of G. Let a_{ij} denote the element in the ith row and jth column of M. Consider

$$\det M = \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) \prod_i a_{i\sigma(i)}.$$

Let $A \in \binom{[m]}{n-1}$, $B = [m] \setminus A$ denote the two disjoint sets $\{\sigma(1), \sigma(2), \ldots, \sigma(n-1)\}$ and $\{\sigma(n), \sigma(n+1), \ldots, \sigma(m)\}$, respectively. For any two such sets, let $\sigma(A, B)$ denote the unique permutation such that $A = \{\sigma(1), \sigma(2), \ldots, \sigma(n-1)\}$, $B = \{\sigma(n), \sigma(n+1), \ldots, \sigma(m)\}$, $\sigma(1) < \sigma(2) < \cdots < \sigma(n-1)$ and $\sigma(n) < \sigma(n+1) < \cdots < \sigma(m)$. Furthermore, let $A(i)$ denote the ith smallest element of A and let $B(i)$ denote the ith smallest element of B. Then we can decompose σ as $\sigma(A, B) \circ \sigma_1 \circ \sigma_2$ where σ_1, σ_2 are permutations on the first $n-1$ elements and last $m-n+1$ elements of σ, respectively. Let $[n, m]$ denote the set $\{n, n+1, \ldots, m\}$. Then summing over all $A \in \binom{[m]}{n-1}$, we have

$$\begin{aligned}
\det M &= \sum_A \sum_{\sigma_1} \sum_{\sigma_2} \mathrm{sgn}(\sigma) \prod_{i=1}^{n-1} a_{i\sigma_1(A(i))} \prod_{j=1}^{m-n+1} a_{(n+j-1)\sigma_2(B(j))} \\
&= \sum_A \mathrm{sgn}(\sigma(A, B)) \left(\sum_{\sigma_1} \mathrm{sgn}(\sigma_1) \prod_{i=1}^{n-1} a_{i\sigma_1(A(i))} \right) \left(\sum_{\sigma_2} \mathrm{sgn}(\sigma_2) \prod_{i=1}^{m-n+1} a_{(n+i-1)\sigma_2(B(j))} \right) \\
&= \sum_A \mathrm{sgn}(\sigma(A, B)) \det M[A, [n-1]] \det M[B, [n, m]].
\end{aligned}$$

Since the incidence matrix of a directed multigraph is totally unimodular, it follows that $\det M[A, [n-1]], \det M[B, [n, m]] \in \{-1, 0, 1\}$, so we have

$$\det M \leq \sum_{A \in \binom{[m]}{n-1}} |\det M[A, [n-1]]| = \sum_{A \in \binom{[m]}{n-1}} (\det M[A, [n-1]])^2.$$

Now by the Cauchy-Binet formula, we have the following equality:

$$\det M_D'^T M_D' = \sum_{S_1} \det M^T[[n-1], S_1] \det M[S_1, [n-1]] = \sum_{S_1} (\det M[S_1, [n-1]])^2.$$

The Kirchhoff matrix-tree theorem implies that $\det M_D'^T M_D' = \mathrm{sp}(G)$, so we have shown the desired inequality. $\square$

This theorem shows that given that one of the incidence submatrices involved in the concatenation is the truncated incidence matrix of a particular planar multigraph G, the construction given in the statement of Theorem 3.1 creates the largest possible determinant out of all choices for the other incidence matrix.

Corollary 3.4. *For any connected planar multigraph G, we have* $\max\det G = \mathrm{sp}(G)$.

4 Bounding the Determinant on Nonplanar Graphs

Given that we can construct the upper bound of Theorem 3.3 for all planar graphs, one might expect there to be a similar construction for nonplanar graphs as well that may be more complex. It turns out that this is not possible. We start by proving a few lemmas.

Lemma 4.1. *Let $c_1, c_2, c_3, \dots, c_k$ be the columns of a incidence submatrix M. Then we have that $c_1, c_2, \dots, c_{k-2}, c_{k-1} + c_k$ form the columns of another incidence submatrix.*

Proof. Let the jth element of column i of the original matrix be denoted c_{ij}. We proceed to show that the elements of the jth row of elements of the new matrix, consisting of $\{c_{ij} | 1 \le i \le k-2\} \cup \{c_{(k-1)j} + c_{kj}\}$, has at most one 1, one -1, and the remaining elements are all equal to 0. We perform casework on the jth row for all $j \in [m]$:

Case 1. *One of $c_{(k-1)j}$ and c_{kj} is equal to* 0.

Since we've deleted one 0 from row j, the total number of 1 and -1 entries remains unchanged, so the desired condition holds.

Case 2. *Both $c_{(k-1)j}$ and c_{kj} are nonzero.*

In this case, since we must have that $c_{(k-1)j} = -c_{kj} = \pm 1$, adding columns $k-1$ and k deletes both nonzero entries from row j and replaces them with zero. Hence the desired condition still holds.

Thus every row of the matrix formed from $c_1, c_2, \dots, c_{k-2}, c_{k-1}+c_k$ has entries in $\{-1, 0, 1\}$ and at most one 1 and at most one -1, so it is an incidence submatrix. □

Lemma 4.2. *Let A be an incidence submatrix such that the last row of A contains only* 0*s, and let B be the incidence submatrix created from removing the last row of A. Then the maximum value of* $\det[A|M]$ *over all incidence submatrices M is at most the maximum value of* $\det[B|N]$ *over all incidence submatrices N.*

Proof. Suppose A is an $m \times (n-1)$ matrix. Let $K = [A|M]$, and let k_{ij} denote the (i,j)-entry of K. We show that there always exists $K' = [B|N]$ such that $|\det K'| \ge |\det K|$. We perform casework based on the number of nonzero entries in the mth row of K.

Notice that $k_{mi} = 0$ for all $i \le n-1$, so it follows that there is at most one 1 and at most one -1 in row m. We divide into cases based on the number of nonzero entries in row m.

Case 1. *There are no nonzero entries in row m.*

All entries in row m must equal 0 so $\det K = 0$. It follows that any choice of N satisfies $|\det K'| \geq |\det K|$ as desired.

Case 2. *There is exactly one nonzero entry in row m.*

Suppose that the nonzero entry appears in column s. Then by Laplace expansion we have

$$\det K = \sum_{j=1}^{m} (-1)^{1+j} k_{mj} \det K_{1,j} = (-1)^{m+s} k_{ms} \det K_{m,s} \leq |\det K_{m,s}|.$$

Notice that no elements of A are in column s since $s \geq n$, so the leftmost $n-1$ columns of $K_{m,s}$ correspond to B. Furthermore, no new 1s or -1s are added to each row of K in $K_{m,s}$, so the rightmost $m-n$ columns of $K_{m,s}$ form an incidence submatrix. Taking $K' = K_{m,s}$ completes the proof.

Case 3. *There are exactly two nonzero entries in row m.*

Let the 1 and -1 entries appear in columns a and b respectively. Since swapping columns does not affect the absolute value of the determinant, without loss of generality suppose $a = m, b = m-1$. Denote the ith column vector of K as c_i. The column vectors $c_n, c_{n+1}, \ldots, c_{m-2}, c_{m-1} + c_m$ form an incidence submatrix by Lemma 4.1. Let K' be obtained by adding the mth column of K to its $(m-1)$th column. We can perform Laplace expansion on K' as follows:

$$\det K = \det K' = \sum_{j=1}^{m} (-1)^{1+j} k_{mj} \det K'_{m,j} = (-1)^{2m} k_{mm} \det K'_{m,m} = \det K'_{m,m}.$$

Notice that the rightmost $m-n$ columns of $K'_{m,m}$ are exactly $c_n, c_{n+1}, \ldots, c_{m-1} + c_m$ with row m removed. Since $c_n, c_{n+1}, \ldots, c_{m-1} + c_m$ form an incidence submatrix, the rightmost $m-n$ columns of $K_{m,m}$ also form an incidence submatrix, so taking $K' = K_{m,m}$ completes the proof.

Thus we have shown the desired result in all cases. □

Remark 4.3. *In the above proof, N can always be constructed by taking column sum operations or removing a column from M. Thus if M has no nonzero entries in a given row i, we can construct N such that $|\det[A|M]| \leq |\det[B|N]|$ and N has no nonzero entries in row i.*

This result additionally shows that any extra rows of zeroes in the left submatrix do not affect the maximal determinant which can be achieved.

Lemma 4.4 (Edgecut Lemma)**.** *Given a multigraph G and an edge e of G, we have that $\max\det G \leq \max\det G^e_{\text{merge}} + \max\det G^e_{\text{cut}}$.*

Proof. Suppose G has m edges and n vertices. Let $\max\det G$ be obtained by the $m \times m$ matrix $[M'_D|N]$, where N is an incidence submatrix and D is an orientation of G. Without loss of generality, suppose the row of M'_D corresponding to edge e is the first row. Denote the entry of $[M'_D|N]$ in the ith row and jth column as a_{ij}. Denote the ith row vector of $[M'_D|N]$ as r_i.

Let $r_{1,\text{merge}} = (a_1, a_2, \ldots, a_m)$ and $r_{1,\text{cut}} = (b_1, b_2, \ldots, b_m)$ where

$$a_i = \begin{cases} a_{1i} & \text{if } i \le n-1 \\ 0 & \text{if } i \ge n \end{cases} \quad \text{and} \quad b_i = \begin{cases} 0 & \text{if } i \le n-1 \\ a_{1i} & \text{if } i \ge n. \end{cases}$$

Then we have $r_1 = r_{1,\text{merge}} + r_{1,\text{cut}}$. Let $M_{\text{merge}} = [r_{1,\text{merge}}\ r_2\ \cdots\ r_m]$ and $M_{\text{cut}} = [r_{1,\text{cut}}\ r_2\ \cdots\ r_m]$. Since the determinant is a multilinear function with respect to its row vectors, we have that $\det M = \det M_{\text{merge}} + \det M_{\text{cut}}$.

We wish to show that $\det M_{\text{merge}} \le \max \det G^e_{\text{merge}}$, so we divide into three cases depending on the number of nonzero entries in $r_{1,\text{merge}}$.

Case 1. *There are no nonzero entries in $r_{1,merge}$.*

Since $r_{1,\text{merge}} = 0$, we must have $\det M_{\text{merge}} = 0$.

Case 2. *There is exactly one nonzero entry in $r_{1,merge}$.*

We can perform Laplace expansion along the first row of M_{merge}. Let the nonzero entry appear in column s, then we have

$$\det M_{\text{merge}} = \sum_{j=1}^{m} (-1)^{1+j} a_{1j} \det M_{1,j} = (-1)^{1+s} a_{1s} \det M_{1,s} \le |\det M_{1,s}|.$$

In particular, the remaining $n-2$ columns of $M_{1,s}$ correspond to columns in M'_D, so they form an incidence submatrix A. Let B be the incidence submatrix obtained by removing all of the rows of zeroes from A, then $\det M_{\text{merge}} \le |\det M_{1,s}| \le \max \det[A|N_1] \le \max \det[B|N_2]$ over all incidence submatrices N_1, N_2 by repeated application of Lemma 4.2 on the empty rows of $M_{1,s}$.

Now we claim that B is exactly the truncated incidence submatrix of G^e_{merge}. In particular, for each edge $e' \in G$ such that $e' \ne e$ such that e has a nonzero entry in column s, if the edge was incident to both v_0 and v_s, then it is deleted since its row becomes a row of 0s. Otherwise, if its head was at v_s, then the new head of the edge becomes v_0. On the other hand, if its tail was at v_s, then the new tail of the edge becomes v_0. All other edges remain unchanged, and this is exactly the construction of G^e_{merge}.

Thus it follows that $\max \det[B|N] = \max \det G^e_{\text{merge}}$, which completes the proof.

Case 3. *There are exactly two nonzero entries in $r_{1,merge}$.*

Suppose the 1 and -1 entries in row 1 appear in columns a and b respectively. Without loss of generality, suppose $a = n-1$, $b = n-2$.

Then let M'_{merge} be the matrix created by adding column $n-1$ to column $n-2$ in M_{merge}. Let the ith column vector of M_{merge} be denoted by c_i. It follows that the leftmost $n-2$ columns of M'_{merge} are exactly $c_1, c_2, \ldots, c_{n-3}, c_{n-2} + c_{n-1}$. By Lemma 4.1, these column vectors form an incidence submatrix. Furthermore, the rows of all zeroes in this new submatrix are exactly those rows such that $c_{n-2} = -c_{n-1} \neq 0$ in M_{merge}. Let $M'_{i,j}$ denote the minor of M'_{merge} created by removing the ith row and jth column and let a'_{ij} be the entry of M'_{merge} in row i and column j. By Laplace expansion along the first row of M'_{merge}, we have

$$\det M_{\text{merge}} = \det M'_{\text{merge}} = \sum_{j=1}^{m} (-1)^{1+j} a'_{1j} \det M'_{1,j} \leq |\det M'_{1,n-1}|.$$

Notice that the leftmost $n-2$ columns of $M'_{1,n-1}$ still form an incidence submatrix A. Let B be obtained by removing the rows of zeroes from A. We have $\det M_{\text{merge}} \leq |\det M'_{1,n-1}| \leq \max \det[A|N_1] \leq \max \det[B|N_2]$ over all incidence submatrices N_1, N_2 by repeated application of Lemma 4.2.

We claim that B is exactly the truncated incidence matrix of G_{merge}. First, for each edge $e' \in G$ such that $e' \neq e$, if e' is not incident to v_{n-1} then it is unaffected. If e' between v_{n-2} and v_{n-1}, then it is mapped to a row of zeroes in $M'_{1,n-1}$, so it is deleted in B. Otherwise, if e' has its head at v_{n-1}, then the new head of e' is mapped to v_{n-2} in M'_{merge}. Similarly, if e' has its tail at v_{n-1} then the new tail of e' is mapped to v_{n-2} in M'_{merge}. This is exactly the construction for the desired contraction, proving the claim.

Hence it follows that $\max \det[B|N_2] = \max \det G^e_{\text{merge}}$, showing the desired claim in all cases.

Now, notice that the leftmost $n-1$ columns of M_{cut} are exactly the leftmost $n-1$ columns of M, except row 1 becomes a row of zeroes. By Lemma 4.2, we can remove this row to obtain an incidence submatrix A such that $|\det M_{\text{cut}}| \leq |\max \det[A|N]|$ over all incidence submatrices N. In particular, since only the row corresponding to edge e is removed from M'_D to obtain A, it follows that A is exactly the truncated incidence matrix of G^e_{cut}, so $\det M_{\text{cut}} \leq \max \det G_{\text{cut}}$.

Thus, it follows that $\max \det G = \det M = \det M_{\text{merge}} + \det M_{\text{cut}} \leq \max \det G^e_{\text{merge}} + \max \det G^e_{\text{cut}}$, completing the proof.

□

Theorem 4.5. *For any nonplanar multigraph G, we have that* $\max \det G \leq \operatorname{sp}(G) - 18$.

Proof. One can check using brute force that the value of $\max \det K_{3,3} = 63$ and the value of $\max \det K_5 = 100$.

Recall that we have that for any multigraph G, we have that $\operatorname{sp}(G) = \operatorname{sp}(G^e_{\text{merge}}) + \operatorname{sp}(G^e_{\text{cut}})$. Let us define the *excess* $\operatorname{exc}(G)$ of a multigraph G to be equal to $\operatorname{sp}(G) - \max \det G$. It follows from Lemma 4.4 that

$$\begin{aligned}\operatorname{exc}(G) &= \operatorname{sp}(G) - \max \det G \\ &\geq \left(\operatorname{sp}(G^e_{\text{merge}}) - \max \det G^e_{\text{merge}}\right) + \left(\operatorname{sp}(G^e_{\text{cut}}) - \max \det G^e_{\text{cut}}\right) \\ &= \operatorname{exc}(G^e_{\text{merge}}) + \operatorname{exc}(G^e_{\text{cut}})\end{aligned}$$

and from Theorem 3.3 that $\operatorname{exc}(G) \geq 0$ for all multigraphs G and edges $e \in G$.

It follows that $\operatorname{exc}(G) \geq \operatorname{exc}(G^e_{\text{merge}})$ and $\operatorname{exc}(G) \geq \operatorname{exc}(G^e_{\text{cut}})$. Thus the excess of a multigraph G is at least that of any multigraph that can be obtained from a sequence of edge contractions and deletions on G. Thus, since Wagner's theorem asserts that one can obtain either $K_{3,3}$ or K_5 from a series of edge contractions and deletions on any nonplanar multigraph G, it follows that $\operatorname{exc}(G) \geq \min\{\operatorname{exc}(K_{3,3}), \operatorname{exc}(K_5)\} = 18$, giving the desired result. □

This result shows that we cannot find a similar construction for nonplanar multigraphs, which shows that a determinant equal to the number of spanning trees cannot be achieved for nonplanar multigraphs. However, it does not rule out the possibility that a large nonplanar multigraph could have a large number of spanning trees with a positive but small excess, resulting in a bigger determinant than the maximal planar multigraph. We rule out this possibility for a special class of graphs:

Theorem 4.6. *If G is a subdivision of $K_{3,3}$ or K_5 with m edges, then* $\max\det G \leq \max \mathrm{sp}_m^{\mathrm{planar}}$.

The proof for this theorem is in Appendix C.

We also hope to generalize this type of result to arbitrary nonplanar multigraphs, which would prove that the full equality holds:

Conjecture 4.7. For all positive integers m, we have $\max\det_m = \max \mathrm{sp}_m^{\mathrm{planar}}$.

5 Asymptotic Bounds

In this section, we study asymptotic behaviors of $\max\det_m$ and $\max \mathrm{sp}_m^{\mathrm{planar}}$ as functions of m. In particular, we give several lower and upper bounds on these two quantities.

5.1 Lower Bounds

First, we give the following lower bound on $\max\det_m$, which is tight for $m \in [6] \cup \{8\}$.

Theorem 5.1. *For all $m \in \mathbb{N}$, there exists an $m \times m$ matrix M that is the concatenation of two incidence submatrices with*

$$\det M = \begin{cases} F_{m+1} & \text{if } m \text{ is odd} \\ F_{m+1} + F_{m-1} - 2 & \text{if } m \text{ is even,} \end{cases}$$

where F_i denotes the ith Fibonacci number.

Proof. Throughout this proof, scripts are taken modulo m. Consider the even case first. We define $M = (a_{ij})_{i,j}$ where

$$a_{ij} = \begin{cases} 1 & \text{if } j \equiv i - 1 \pmod{m} \\ 1 & \text{if } j \equiv i \pmod{m} \\ -1 & \text{if } j \equiv i + 1 \pmod{m} \\ 0 & \text{otherwise.} \end{cases}$$

Since exchanging two columns preserves the absolute value of the determinant, the sets of odd-indexed and even-indexed columns partition M into two incidence submatrices of M.

We show that $\det M = F_{m+1}+F_{m-1}-2$. We use the formula $\det M = \sum_\sigma \operatorname{sgn}(\sigma) \prod_i a_{i\sigma(i)}$. Fix a permutation σ corresponding to a nonzero term of the summation. Mark each row i with $\ell_i = \mathrm{U}, \mathrm{C}, \mathrm{L}$ depending on whether $\sigma(i) \equiv i+1, i, i-1 \pmod{m}$, respectively.

We now perform casework on the permutation σ. If $\ell_i = \ell_{i+1} = \mathrm{U}$ for some $i \in [m]$ it follows that $\ell_i = \mathrm{U}$ for all i and $\sigma = (1\ 2\ 3\ \cdots\ m)$. Similarly, if $\ell_i = \ell_{i+1} = \mathrm{L}$ for some $i \in [m]$, then $\ell_i = \mathrm{L}$ for all i. These two cases contribute $\operatorname{sgn}(1\ 2\ 3\ \cdots\ m)(-1)^m = -1$ and $\operatorname{sgn}((1\ 2\ 3\ \cdots\ m)^{-1})1^m = -1$ to the determinant, respectively.

For all the remaining permutations, if $\ell_i = \mathrm{U}$ then $\sigma(i) = i+1$ so $\ell_{i+1} \neq \mathrm{C}$. Since there are no two Us in a row, we have $\ell_{i+1} = \mathrm{L}$. Conversely, if $\ell_i = \mathrm{L}$ then $\sigma(i) = i-1$ so $\ell_{i-1} \neq \mathrm{C}$. Thus $\ell_{i-1} = \mathrm{U}$.

Hence σ consists of blocks of C and blocks of UL in a cycle of m letters. In particular, for a permutation σ with k blocks of UL, the term contributes $\operatorname{sgn}(\sigma)(-1)^k$ to the determinant. However, since σ can be written as the product of k adjacent transpositions, with one between the two elements of each UL block, it follows that $\operatorname{sgn}(\sigma) = (-1)^k$, so the permutation contributes exactly 1 to the determinant.

Finally, to count the total number of such blocks, we proceed with casework on whether 1 and n are together in a UL block. If not, the problem becomes equivalent to the number of ways to partition n consecutive numbers into consecutive blocks of sizes 1 and 2, which is equal to F_{n+1} (this can be proved either by an inductive argument or by generating functions). Otherwise, the problem becomes equivalent to the number of ways to partition the $n-2$ remaining numbers $2, 3, \ldots, n-1$ into consecutive blocks of sizes 1 and 2, which is equal to F_{n-1}. Thus, adding all of the cases show that $\det M = F_{n+1} + F_{n-1} - 2$.

For the odd case, we instead define

$$a_{ij} = \begin{cases} 1 & \text{if } j = i-1 \\ 1 & \text{if } j = i \\ -1 & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Again, the sets of odd-indexed and even-indexed columns partition M into two incidence submatrices of M.

We now define $\ell_i = \mathrm{U}, \mathrm{C}, \mathrm{L}$ for each row i in the same way, with the only difference being that $\ell_1 \neq \mathrm{L}$ and $\ell_m \neq \mathrm{U}$. The only permutations which are removed from the cases are the case where all $\ell_i = \mathrm{L}$, the case where all $\ell_i = \mathrm{U}$, and the case when $n, 1$ are in a UL block together. It still holds that each permutation formed from C and UL blocks contributes 1 to the determinant. Thus, we have $\det M = F_{n+1}$. □

Corollary 5.2. *We have that* $\max \det_m = \Omega(\phi^m)$ *where* $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

In combinatorial optimization, one might be interested in *fractionality* of extreme points of a polyhedron. For instance, Pritchard [6] showed that there exist extreme points of the

Held-Karp relaxation (also known as the subtour elimination relaxation) of the symmetric traveling salesman problem with fractionality exponentially small in the number of vertices. We show that the minimal fractionality of an extreme point of a polyhedron $P = \{x : Mx \leq b\}$ where M is the concatenation of two incidence submatrices and b is an integral vector can be exponentially small in the dimension of M by slightly modifying the above construction.

Theorem 5.3. *The minimum fractionality of a vector $x = M^{-1}b$ over all invertible matrices M which can be written as the concatenation of two incidence submatrices and over all integral vectors b is at most $\frac{1}{F_{n+1}}$.*

The proof of this theorem proceeds similarly to that of Theorem 5.1 and is in Appendix D.

We can also use Theorem 3.1 to obtain lower bounds on $\max\det_m$ from lower bounds on $\max \mathrm{sp}_m^{\mathrm{planar}}$. In particular, the bound from Theorem 5.1 can also be obtained by using the directed planar dual construction on the wheel graph with m edges. Furthermore, a classical result from statistical physics on the spanning tree entropy of square lattices implies a better lower bound on $\max \mathrm{sp}_m^{\mathrm{planar}}$.

Let C denote Catalan's constant $\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$. Let $\mathcal{L}(\mathbb{Z}^2)$ denote the regular square lattice, which has vertex set $\mathbb{Z}$ and rook-adjacent edges. A *grid graph* is defined to be a finite connected subgraph of $\mathcal{L}(\mathbb{Z}^2)$ and is always planar.

Theorem 5.4 (Wu, 1977, [9])**.** *For all $m \in \mathbb{N}$, there exists a grid graph G with m edges such that* $\mathrm{sp}(G) = \Omega(e^{2Cm/\pi})$.

Corollary 5.5. *We have that* $\max\det_m = \Omega(e^{2Cm/\pi})$ *where* $e^{2C/\pi} \approx 1.791$.

5.2 Upper Bounds

Unlike the lower bounds, upper bounds on $\max \mathrm{sp}_m^{\mathrm{planar}}$ cannot be directly transformed into upper bounds on $\max\det_m$ without proving Conjecture 4.7. In this subsection, we provide several upper bounds on $\max\det_m$, which automatically give upper bounds on $\max \mathrm{sp}_m^{\mathrm{planar}}$. First, we have the following trivial upper bound:

Theorem 5.6. *For any positive integer m, we have that* $\max\det_m \leq \binom{m}{\lfloor m/2 \rfloor}$.

Proof. From Theorem 3.3, we know that $\max\det_m \leq \mathrm{sp}(G)$ over all multigraphs G with m edges. Suppose G has n vertices, then since each spanning tree consists of exactly $n-1$ edges, we have that $\mathrm{sp}(G) \leq \binom{m}{n-1}$. Since $\max_n \binom{m}{n-1} = \binom{m}{\lfloor m/2 \rfloor}$, we obtain the desired result. $\square$

Since $\binom{m}{\lfloor m/2 \rfloor} \sim 2^m$, this is an exponential upper bound. According to a user JimT on Mathematics Stack Exchange, he claimed to have found an elementary proof that $\max \mathrm{sp}_m^{\mathrm{planar}} \leq 1.93^m$.[3] We were unable to find their proof online or in the literature, but we were able to obtain an exponentially stronger result by bounding the number of nonzero entries in the matrix:

[3] See `https://math.stackexchange.com/questions/2832917/maximum-number-of-spanning-trees-of-a-planar-graph-with-a-fixed-number-of-edges/3969689`.

Theorem 5.7. *For any positive integer m, we have that* $\max\det_m \leq (\sqrt[3]{7})^m \approx 1.913^m$.

Proof. We proceed by strong induction. Note that the result holds for $m = 1$ trivially. Given that the claim is true for all $m = 1, 2, \ldots, k-1$, we show it is true for $m = k$.

Suppose $\max\det_m$ is achieved by the absolute value of the determinant of the matrix $M = [M'_D | M'_F] = (a_{ij})_{i,j}$ where D, F are orientations of multigraphs G, H, respectively, each with m edges. Let n be the number of vertices of G. Consider the $m \times (m+2)$ rectangular matrix $[M_D | M_F]$. This matrix has at most $4m$ nonzero entries since each row has at most two nonzero entries in M_D, M_F respectively. Since there are $m+2$ columns, there exists a column with at most $\lfloor \frac{4m}{m+2} \rfloor = 3$ nonzero entries. If this column is a column of $[M'_D | M'_F]$ then M has a column with at most three nonzero entries.

Otherwise, without loss of generality suppose the column is the truncated column of the incidence matrix of G. Let $c_1, c_2, \ldots, c_m$ be the column vectors of M. Since the sum of the column vectors of M_D is equal to 0, it follows that the truncated column c_0 is equal to $-1 \cdot (c_1 + c_2 + \cdots + c_{n-1})$. It follows that

$$\begin{aligned}
\det M &= \det \begin{bmatrix} c_1 & c_2 & \ldots & c_m \end{bmatrix} \\
&= \det \begin{bmatrix} c_1 + c_2 + \cdots + c_{n-1} & c_2 & \ldots & c_m \end{bmatrix} \\
&= -\det \begin{bmatrix} c_0 & c_2 & \ldots & c_m \end{bmatrix}.
\end{aligned}$$

Thus, we can choose a different matrix M' formed by replacing column c_1 with c_0 in M such that $|\det M'| = \det\max_m$ and such that there is a column of M' with at most three nonzero entries. Replacing M with M', we can assume M has a column with at most three nonzero entries.

Since we can switch the order of the rows and columns without affecting the absolute value of the determinant, assume without loss of generality that the first column of M has at most three nonzero entries and these $k \leq 3$ nonzero entries appear in the rows $1, 2, \ldots, k$. Let r_i denote the ith row vector of M, and let $r_{1,0} = (a_1, a_2, \ldots, a_m)$ and $r_{1,1} = (b_1, b_2, \ldots, b_m)$ where

$$a_i = \begin{cases} a_{1i} & \text{if } i \leq n-1 \\ 0 & \text{if } i \geq n \end{cases} \quad \text{and} \quad b_i = \begin{cases} 0 & \text{if } i \leq n-1 \\ a_{1i} & \text{if } i \geq n. \end{cases}$$

Analogously define $r_{2,0}, r_{2,1}, r_{3,0}$, and $r_{3,1}$. Then from the multilinearity of the determinant, we have

$$\det M = \sum_{i,j,k \in \{0,1\}} \det \begin{bmatrix} r_{1,i} & r_{2,j} & r_{3,k} & r_4 & \ldots & r_m \end{bmatrix}.$$

However, since the matrix formed by $r_{1,1}, r_{2,1}, r_{3,1}, r_4, \ldots, r_m$ has no nonzero entries in its first column, it has determinant 0. Furthermore, for any of the other matrices in the summation, each row has the property that it either has no nonzero entries in M'_A or has no nonzero entires in M'_B, so it can be removed using Lemma 4.2 without decreasing the absolute value of the determinant and without losing the same property in the other rows by Remark 4.3.

Thus for all $i, j, k \in \{0, 1\}$ and $(i, j, k) \neq (1, 1, 1)$ we can apply Lemma 4.2 to the first three rows in succession to obtain a reduction to an $(m-3) \times (m-3)$ matrix $N_{i,j,k}$ such that $N_{i,j,k}$ can be written as the concatenation of two incidence submatrices and $|\det N_{i,j,k}| \geq |\det(r_{1,i}, r_{2,j}, r_{3,k}, r_4, \ldots, r_m)|$. Hence,

$$\det M = \sum_{i,j,k \in \{0,1\}, (i,j,k) \neq (1,1,1)} N_{i,j,k} \leq 7 \cdot (\sqrt[3]{7})^{m-3} = (\sqrt[3]{7})^m,$$

as desired.

□

6 Conclusion and Future Directions

In Theorem 3.1, we establish an interesting connection between $\max \det_m$ and $\max \mathrm{sp}_m^{\mathrm{planar}}$. In addition, we provide some evidence supporting Conjecture 4.7 that the reverse direction also holds, including Theorems 4.5 and 4.6, and Table 1 in the appendix. If this conjecture holds, then it implies an algebraic characterization for computing the maximum number of spanning trees in a planar multigraph.

There are several potential approaches to prove Conjecture 4.7. One possible approach is to devise an algorithm that transforms a square matrix $[M'_{D_1} | M'_{D_2}]$ with D_1 and D_2 nonplanar to some other square matrix $[M'_{D_1} | M'_{D'_2}]$ such that D'_2 has fewer spanning trees than D_2, while preserving the determinant. Another possible method is to extend Theorem 4.6 to show that any nonplanar graph G underperforms some planar graph with the same number of edges in terms of $\max \det G$. In addition, the application of Lemma 4.4 in the proof of Theorem 4.5 is extremely loose. For most nonplanar graphs, one can obtain many copies of K_5 or $K_{3,3}$ as minors, which could potentially be used to prove stronger versions of Theorem 4.5.

For asymptotic bounds on $\max \det_m$ and $\max \mathrm{sp}_m^{\mathrm{planar}}$, we have established that $c_1 1.791^m \leq \max \mathrm{sp}_m^{\mathrm{planar}} \leq \max \det_m \leq c_2 1.913^m$ for some constants $c_1, c_2 > 0$. It would be interesting to improve these bounds and to close the gap. Matching lower and upper bounds would also imply Conjecture 4.7.

Acknowledgments

I'd like to thank my mentor, Yuchong Pan, for providing the guidance, support, and the resources that led to this research project. I'd like to thank Prof. Michel X. Goemans for proposing my research topic and his talks on combinatorial optimization. I'd like to thank Dr. Tanya Khovanova for her invaluable advice along the way. I'd like to thank Peter Gaydarov, Allen Lin, and an anonymous reviewer for their helpful suggestions. I'd like to thank RSI, CEE, and MIT for making all of this possible. I'd also like to thank all of the sponsors of RSI.

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A Exact Values of $\max\det_m$, $\max\mathrm{sp}_m^{\mathrm{planar}}$ and $\max\mathrm{sp}_m^{\mathrm{general}}$

m	$\max\det_m$	$\max\mathrm{sp}_m^{\mathrm{planar}}$	$\max\mathrm{sp}_m^{\mathrm{general}}$
1	1	1	1
2	2	2	2
3	3	3	3
4	5	5	5
5	8	8	8
6	16	16	16
7	24	24	24
8	45	45	45
9	75	75	$\underline{81}$
10	130	130	$\underline{135}$

Table 1: Values of $\max\det_m$, $\max\mathrm{sp}_m^{\mathrm{planar}}$ and $\max\mathrm{sp}_m^{\mathrm{general}}$ for $m \in [10]$.

B Alternative Proof of Theorem 3.1

Proof. Let $\mathcal{C}(G)$ be a cycle basis for the circulation space of G. Since G is planar, by Mac Lane's planarity criterion there exists a 2-basis K for $\mathcal{C}(G)$. For each cycle in this basis, we can define a fixed direction of flow around the cycle such that every vertex is the source of one edge flow and the sink of another edge flow of the cycle. In particular, if G has m edges and n vertices then K can be written as an $m \times (m-n+1)$ matrix where each column corresponds to a basis element and an entry in the row corresponding to a given directed edge of the chosen orientation D of G is 1 if it points along the flow of the basis element, -1 if it points against the flow, and 0 if it does not belong to the basis element.

Let $c_1, c_2, \ldots, c_{n-1}$ be the column vectors of M'_D and let $d_1, d_2, \ldots, d_{m-n+1}$ be the columns of K. Importantly, we know that K is an orthogonal complement of G since for any $i \in [n-1], j \in [m-n+1]$, the value of $c_i \cdot d_j$ represents the total amount of accumulation of the flow corresponding to d_j at vertex i, which is always equal to 0 for cycles. As in the previous proof,

$$\begin{aligned}
\det\left[\begin{array}{c|c} M'_D & K \end{array}\right]^2 &= \det\left[\begin{array}{c|c} M'_D & K \end{array}\right]^T \left[\begin{array}{c|c} M'_D & K \end{array}\right] \\
&= \det\begin{bmatrix} M'^T_D \\ K^T \end{bmatrix} \left[\begin{array}{c|c} M'_D & K \end{array}\right] \\
&= \det\begin{bmatrix} M'^T_D M'_D & 0 \\ 0 & K^T K \end{bmatrix} \\
&= \det M'^T_D M'_D \cdot \det K^T K.
\end{aligned}$$

Linearly independent sets of rows of K correspond to linearly independent edges in the directed multigraph D' which has K as its incidence submatrix, and $\det M'^T_{D'} M'_{D'}$ is exactly the number of such sets (i.e. the number of spanning trees of D'). Hence it suffices to show that there is a bijection between the spanning trees of G and the linearly independent sets of K in order to show that $\det K^T K = \det M'^T_D M'_D$.

In K, I claim that the rows corresponding to some set of $r = m + 1 - n$ edges are linearly independent if and only if we can remove any $r - 1$ of them from the graph, and the last remaining edge is always part of a cycle. This claim is equivalent to the assertion that for each of those r edges, there is an element of the column space of K such that the entries of the other $r - 1$ rows are 0 and not the specified edge.

It is clear that this is algebraically equivalent to the linearly independence of those r rows of K. Furthermore, looking at the edges of G which are not one of these r edges, this condition becomes equivalent to enumerating the sets of $n - 1$ edges of G with the property that the addition of any other edge of G creates a cycle containing the new edge in the resulting subgraph. I claim that this holds if and only if those $n - 1$ edges form a spanning tree. If they do not form a spanning tree then the resulting subgraph of G obtained by picking an edge between any two disjoint components would violate the condition. On the other hand, if the $n - 1$ edges form a spanning tree, then there is a path between any two vertices of G, so simply combining the new edge with the path between the endpoints of the new edge gives the desired cycle. Hence the bijection holds, so it follows that K has the same number of linearly independent sets as spanning trees of G, which finishes. □

C Proof of Theorem 4.6

Lemma C.1. *Let A be a $m \times m$ matrix with row vectors $r_1, r_2, \ldots, r_m$. If the first column vector of A is equal to $(1, -1, 0, 0, \cdots, 0)$ then $\det A$ is equal to the determinant of the $(m-1) \times (m-1)$ matrix formed by the rows $r_1 + r_2, r_3, r_4, \ldots, r_m$ with their first entries removed.*

Proof. Let $A = (a_{ij})_{i,j}$. Let the matrix B be obtained by adding the first row to the second row of A. The only nonzero entry in the first column of B is $a_{11} = 1$. We can perform Laplace expansion along the first column of B as follows:

$$\det A = \sum_{i=1}^{m} (-1)^{i+1} k_{i1} \det K_{i,1} = (-1)^{1+1} k_{11} \det K_{1,1} = \det K_{1,1}.$$

Notice that $K_{1,1}$ is exactly the $(m-1) \times (m-1)$ matrix formed by the rows $r_1 + r_2, r_3, r_4, \ldots, r_m$ with their first entries removed, finishing. □

Now we have the proof of Theorem 4.6:

Proof. First, we consider the case when G is a subdivision of $K_{3,3}$. For each edge e of the original $K_{3,3}$, let G^e denote a tuple of edges $(G^e_1, G^e_2, \ldots, G^e_k)$ such that $G^e_1, \ldots, G^e_k$ are the edges created from subdivisions on the original edge e and such that G^e_i is incident to G^e_{i+1}

for all $i \in [k-1]$. Since $K_{3,3}$ has 6 vertices and 9 edges, and since each subdivision adds a vertex and an edge, it follows that G has $m-3$ vertices. Suppose that $\max\det G$ is attained by the matrix $M = [M'_D|N] = (a_{ij})_{i,j}$ where N is the truncated incidence matrix of another multigraph G' and D is some orientation of G. Without loss of generality, choose the orientation so that the head of G_i^e is the tail of G_{i+1}^e for all $i \in [k-1]$. Then G' has $m+2-(m-3) = 5$ vertices.

Now each pair of edges G_i^e, G_{i+1}^e in G, there exists a vertex $v \in G$ such that $\deg v = 2$ and v is incident to G_i^e, G_{i+1}^e. We call this vertex the ith *child* of edge e. Let this vertex correspond to column i in M'_D, and let the columns of M be $c_1, c_2, \ldots, c_m$. Let n be the number of vertices of G. If column i is the truncated column, then since the sum of the columns of M_D is 0, we know $c_0 = -1\cdot(c_1+c_2+\cdots+c_{n-1})$ is the column corresponding to the missing vertex. We can then replace M with $\begin{bmatrix} c_0 & c_2 & c_3 & \cdots & c_m \end{bmatrix}$, whose determinant has the same absolute value since it can be written as the composition of column sum operations and scaling columns by -1. Thus we can assume this column is not the truncated column and can freely choose which vertex corresponds to the truncated column.

Now for each edge e of the original $K_{3,3}$, let the truncated column be a vertex of the original $K_{3,3}$ which is not an endpoint of e. Suppose k is the number of elements of G^e. Rearrange the columns of M'_D so that the ith child of e is in the ith column for all $i = [k-1]$. Then rearrange the rows of M'_D so that G_i^e lies in the ith row for all $i \in [k]$. Let

$r_{1,0} = (a_1, a_2, \ldots, a_m)$ and $r_{1,1} = (b_1, b_2, \ldots, b_m)$ where

$$a_i = \begin{cases} a_{1i} & \text{if } i \le n-1 \\ 0 & \text{if } i \ge n \end{cases} \quad \text{and } b_i = \begin{cases} 0 & \text{if } i \le n-1 \\ a_{1i} & \text{if } i \ge n. \end{cases}$$

Analogously define $r_{i,0}, r_{i,1}$ for all $i \in m$.

We can now repeatedly apply Lemma C.1 to M for $k-1$ times to obtain a $(m-k+1)\times(m-k+1)$ matrix M'. Note that each time the operation is applied, a child of e is effectively contracted into one of its neighbors, so the first row of M' corresponds to the edge e of the original $K_{3,3}$. Now notice that the first row of the resulting matrix is exactly $\sum_{i=1}^k r_{i,0} + \sum_{i=1}^k r_{i,1}$, which is linear in terms of the sum of all $r_{i,1}$. Since the determinant is a multilinear function, it follows that since M is maximal, that we can assume that $r_{i,1}$ is equal for all i since we are maximizing a linear function on the sum of all $r_{i,1}$. This does not change the left submatrix of M. Thus, by setting $r_{i,1}$ for $i \in [k]$ to be identical in M, we can obtain another matrix $M = [M'_D|N']$ which attains the maximal possible $|\det M|$ such that every edge in G^e has identical entries in submatrix N.

We can then repeat this algorithm for all edges e of the original $K_{3,3}$ to obtain a matrix M with maximal $|\det M|$ and such that there are at most 9 distinct values of $r_{i,1}$: one for each e. Thus it follows that the underlying simple graph of G' has at most 9 edges and 5 vertices, so it must be planar. Hence G' is a planar graph, so we have that $\max\det G = |\det M| \le \operatorname{sp}(G') \le \max \operatorname{sp}_m^{\text{planar}}$ by Theorem 3.3, finishing.

Now, consider the case when G is a subdivision of K_5. Identically to the proof of $K_{3,3}$, we define $G^e = (G_1^e, G_2^e, \ldots, G_k^e)$ for each edge e of the original K_5 with the same properties.

Again, let $\max\det G$ be attained by the matrix $M = [M'_D|N] = (a_{ij})_{i,j}$ where N is the truncated incidence matrix of another multigraph G' and D is some orientation of G. Without loss of generality, choose the orientation so that the head of G_i^e is the tail of G_{i+1}^e for all $i \in [k-1]$. Then G' has $m + 2 - (m - 5) = 7$ vertices.

Using the same application of Lemma C.1 as in the $K_{3,3}$ case, we can choose a matrix M such that for every edge e of the original K_5, all rows i such that their corresponding edge is in G^e have identical values for $r_{i,1}$. We will denote this shared value as r_e. Thus, the underlying simple graph G'' of the multigraph G' whose truncated incidence matrix is N has at most 10 edges on 7 vertices.

If G'' is planar then G' is planar so we have $\max\det G = |\det M| \le \mathrm{sp}(G') \le \max \mathrm{sp}_m^{\mathrm{planar}}$ by Theorem 3.3. Otherwise, G'' must be nonplanar. Furthermore, since $2|E| = \sum_v \deg v$, it follows by the pigeonhole principle that there exists a vertex v with degree at most $\lfloor \frac{20}{7} \rfloor = 2$ in G''. We can then divide into cases based on the degree of this vertex in G''.

If the degree of v in G'' is 0, then its degree in G' is also 0, so it follows that G' is disconnected. Hence $\det M \le \mathrm{sp}(G') = 0$, finishing trivially.

If the degree of v in G'' is 1, then removing v from G'' gives a graph on 6 vertices with at most 9 edges. This graph must be nonplanar, so it must equal exactly $K_{3,3}$. Hence it follows that every edge e of the original K_5 maps to a distinct edge of G''. Let e be the edge of the original K_5 which maps to the only edge incident to v. As in the $K_{3,3}$ case, rearrange the columns of M'_D so that the ith child of e is in the ith column for all $i = [k-1]$ where k is the number of elements of G^e. Then rearrange the rows of M'_D so that G_i^e lies in the ith row for all $i \in [k]$. Define $r_{i,0}$ and $r_{i,1}$ analogously as in the $K_{3,3}$ case.

We can then apply Lemma C.1 to M for $k-1$ times in succession to obtain an $(m-k+1) \times (m-k+1)$ matrix M' such that the entries of the first row of M' in the left submatrix corresponds to the directed edge between the tail of G_1^e and the head of G_k^e, and the entries of the first row of M' in the right submatrix N is equal to $k \cdot r_e$. Notice furthermore that the only nonzero entry in the column of N corresponding to vertex v lies in row 1, so the entries of the first row in the left submatrix do not affect the determinant since they do not appear in the Laplace expansion along the specified column v.

Thus, we can change the head of G_k^e to be another vertex of the original K_5 which is not an endpoint of e without affecting $|\det M|$. This has the effect of changing the left submatrix to become the truncated incidence matrix of a different multigraph H such that H is a subdivision of some multigraph on 5 vertices with 10 edges that is not K_5. This multigraph is necessarily planar, so it follows that $\max\det G \le \det M \le \max\det H \le \mathrm{sp}(H) \le \max \mathrm{sp}_m^{\mathrm{planar}}$ by Theorem 3.3, finishing.

Finally, consider the case when the degree of v in G'' is 2. It follows that G'' is the subdivision of another graph H on 6 vertices and at most 9 edges. Since planarity is preserved under the subdivision operation, H must be $K_{3,3}$ with exactly 9 edges, so no two edges of e map to the same edge in G''. So let e_1, e_2 be the two edges of the original K_5 which map to the two edges incident to v in G''. Since we can freely arrange the columns and rows of M, as long as the columns of M'_D and N are preserved, so we can in fact apply Lemma C.1 on an column with two nonzero entries such that one entry is 1 and the other is -1. Thus, we can repeatly perform Lemma C.1 on all of the children of edges e_1, e_2. Let k_1, k_2 be the number of elements of G^{e_1}, G^{e_2} respectively.

After all of these operations along with row swaps, we obtain a $(m - k_1 - k_2 + 2) \times (m - k_1 - k_2 + 2)$ matrix M' such that the ith row for $i \geq 3$ corresponds to a row of M with the entries corresponding to the children of e_1, e_2 removed. Furthermore, the entries of the first row of M' in the left submatrix correspond to a directed edge from the tail of $G_1^{e_1}$ to the head of $G_{k_1}^{e_1}$, while the entries of the first row of M' in the right submatrix are equal to $k_1 \cdot r_{e_1}$. Similarly, the entries of the second row of M' in the left submatrix correspond to a directed edge from the tail of $G_1^{e_2}$ to the head of $G_{k_2}^{e_2}$, while the entries of the second row of M' in the right submatrix are equal to $k_2 \cdot r_{e_2}$. Let $s_1, s_2, \ldots, s_t$ denote the row vectors of M', where $t = m - k_1 - k_2 + 2$. Define $s_{i,0}, s_{i,1}$ for all i analogously to $r_{i,0}$ and $r_{i,1}$.

We also know that the only two nonzero entries in the column of M' corresponding to the vertex v of G' belong in s_1, s_2. Since we can scale s_1, s_2 by -1 by flipping the direction of all of the edges in G^{e_1}, G^{e_2} in M, respectively, we can assume without loss of generality that the nonzero entries in s_1, s_2 are $k_1, -k_2$ respectively. So we have:

$$\det M' = k_1 \cdot k_2 \cdot \begin{bmatrix} \frac{s_1}{k_1} & \frac{s_2}{k_2} & s_3 & s_4 & \cdots & s_t \end{bmatrix}$$

We can then apply Lemma C.1 on $\begin{bmatrix} \frac{s_1}{k_1} & \frac{s_2}{k_2} & s_3 & s_4 & \cdots & s_t \end{bmatrix}$ so that the determinant of this matrix is equal to the determinant of the matrix formed by $\frac{s_1}{k_1} + \frac{s_2}{k_2}, s_3, s_4, \ldots, s_t$ after removing the entries of each vector corresponding to vertex v of G'. Notice that the vector $\frac{s_1}{k_1} + \frac{s_2}{k_2}$ is linear in $\frac{s_{1,0}}{k_1} + \frac{s_{2,0}}{k_2}$. From the multilinearity of the determinant it follows that if A is the matrix obtained by replacing s_1 with $s_{2,0} + s_{1,1}$ in M' and B is the matrix obtained by replacing s_2 with $s_{1,0} + s_{2,1}$ in M', then

$$\det M' = \frac{\det A + \det B}{2}$$

. So without loss of generality, assume that $|\det A| \geq |\det M'|$.

We can obtain A instead of M' from the same applications of Lemma C.1 on M by modifying M into K by replacing the tail of $G_1^{e_1}$ with the tail of $G_1^{e_2}$ and the head of $G_{k_1}^{e_1}$ with the head of $G_{k_2}^{e_2}$. This changes only the value of $s_{1,0}$ to equal the value of $s_{2,0}$ in the matrix M', which is exactly the change to create matrix A.

Hence $|\det K| = |\det A| \geq |\det M'| = |\det M|$. However, from the construction of K, we know that the multigraph H whose truncated incidence matrix is the left submatrix of K can be expressed as the subdivision of a graph with 5 vertices and 10 edges with some pair of edges sharing the same endpoints. Thus H is the subdivision of a graph which is planar, and must also be planar. Hence we have $\det \max G = |\det M| \leq |\det K| \leq \mathrm{sp}(H) \leq \max \mathrm{sp}_m^{\mathrm{planar}}$ by Theorem 3.3, finishing.

Thus we have exhausted all cases, so we are done.

□

D Proof of Theorem 5.3

Proof. For all $n \in \mathbb{N}$, we define

$$a_{ij} = \begin{cases} 1 & \text{if } j = i - 1 \\ 1 & \text{if } j = i \\ -1 & \text{if } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Fix a permutation σ corresponding to a nonzero term of the summation. Mark each row i with $\ell_i = \mathrm{U}, \mathrm{C}, \mathrm{L}$ depending on whether $\sigma(i) \equiv i + 1, i, i - 1 \pmod{m}$, respectively. We compute the determinant of the minors $M_{1,1}$ and $M_{1,2}$, respectively. The case of $\ell_i = \mathrm{U}$ for all i and the case of $\ell_i = \mathrm{L}$ for all i contribute zero to the summation. Hence, we assume that σ is composed of blocks of C and UL over a cycle of m elements. Note that computing $\det M_{1,1}$ is equivalent to fixing $\ell_1 = \mathrm{C}$, and freely placing rows $2, 3, \ldots, n$ into blocks of sizes 1 and 2, yielding F_n solutions each contributing 1 to the determinant. This shows that $\det M_{1,1} = F_n$. Computing $\det M_{1,2}$ is equivalent to fixing rows $1, 2$ into a UL block, and freely placing rows $3, 4, \ldots, n$ into blocks of sizes 1 and 2, yielding $-F_{n-1}$ for the determinant.

Note that $\gcd(F_n, F_{n-1}) = \gcd(F_{n-1} + F_{n-2}, F_{n-1}) = \gcd(F_{n-2}, F_{n-1})$. By an inductive argument on n, it follows that $\gcd(F_n, F_{n-1}) = \gcd(1, 1) = 1$. By Bezout's theorem, there exist $r, s \in \mathbb{Z}$ such that $rF_n - sF_{n-1} = 1$. Now, let $b \in \mathbb{N}^n$ such that $b_1 = r, b_2 = s$, and $b_k = 0$ for $k \geq 3$. By Cramer's rule, the first coordinate x_1 of the solution to $Mx = b$ satisfies

$$x_1 = \frac{\det(M_1)}{\det(M)} = \frac{rF_n - sF_{n-1}}{F_{n+1}} = \frac{1}{F_{n+1}}.$$

This completes the proof. $\square$