On Small Spherical 2-Distance Sets in n-Dimensional Euclidean Space

Iliyas Noman

Under the direction of
Yuan Yao
PhD Student
Massachusetts Institute of Technology

Research Science Institute
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Abstract

A set of points S in n-dimensional Euclidean space $\mathbb{R}^n$ is called a *2-distance set* if the set of pairwise distances between the points has cardinality two. The 2-distance set is called *spherical*, if its points lie on the unit sphere in $\mathbb{R}^n$. In $\mathbb{R}^n$ there is a finite number of 2-distance sets with $n+2$ points and an infinite number of 2-distance sets on $n+1$ points and fewer. We characterize the spherical 2-distance sets with $n+2$ points in $\mathbb{R}^n$.

Summary

A spherical 2-distance set is a set of points in an Euclidean space such that there are only two distances between the points, and all points lie on a sphere. We determine some properties of the spherical two distance sets, and we characterize all smallest spherical 2-distance sets.

1 Introduction

The problem of sets with few distances is considered to be naturally emerging in discrete geometry and coding theory. One can view *2-distance sets* (a finite set of points in n-dimensional space, allowing only to possible distances between its elements) as a natural next subject of study after equiangular lines, and equiangular lines themselves have numerous applications.

The study of 2-distance sets began in 1947 when Kelly [1] showed that a 2-distance set in the plane can have at most 5 points. In 1963, Croft [2] showed that a 2-distance set in $\mathbb{R}^3$ can have at most 6 points. Mathematicians became interested in solving this problem in the more general case $\mathbb{R}^n$. A natural question that arose is what the size of the maximal 2-distance set is. Larman, Rogers, and Seidel [3] found an upper bound on the size of a 2-distance set in $\mathbb{R}^n$ that is asymptotically tight, thus resolving this question.

The other extreme case, the minimal 2-distance sets, has also been extensively studied. There are infinitely many non-isomorphic 2-distance sets with $n+1$ points or fewer, so the smallest interesting case is $n+2$ points. In 1966, Einhorn and Schoenberg [4] (Theorem 2.1) showed that there is a finite number of 2-distance sets in $\mathbb{R}^n$ with $n+2$ points and gave a combinatorial interpretation for the number of such "small" sets.

Motivated by Theorem 2.1, we are interested in characterizing the 2-distance sets whose points lie on the unit sphere. We call those sets *spherical*. A previous characterization of the spherical 2-distance sets was completed in 2012 by Nozaki and Shinohara [5], who gave a necessary and sufficient condition for a graph to have a representation as a spherical 2-distance set in some Euclidean space based on Roy's previous results [6]. The conditions in [5, Theorem 2.4] involve bulky expressions that use the eigenvalues of the adjacency matrix and the angles of its eigenvectors with the all-ones vector. In contrast, here we focus on the "small" spherical graphs (with exactly $n+2$ vertices in $\mathbb{R}^n$) and obtain a much cleaner necessary and sufficient condition for a graph to have a spherical representation in this case, only using the eigenvalues of the adjacency matrix and the eigenvalues of its projection onto a subspace.

Theorem 1.1. *Let G be a graph on $n+2$ vertices whose adjacency matrix A_G has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+2}$. Let J be the all-ones matrix and $P = I - \frac{1}{n+2}J$ be the projection matrix onto the subspace orthogonal to the all-ones vector. Then G has a spherical representation in $\mathbb{R}^n$ if and only if the maximum eigenvalue of PA_GP is equal to λ_2 and the multiplicity of the eigenvalue λ_2 in A_G (excluding λ_1 if $\lambda_1 = \lambda_2$) is the same as the multiplicity of λ_2 in PA_GP.*

The paper is organized as follows. In Section 2, we define a 2-distance set more formally, introduce the concepts of (spherically) representable graphs, and recall two results from the literature that we use in the paper. In Section 3, we give four initial conditions for a graph on $n+2$ vertices to be representable as a spherical 2-distance set in $\mathbb{R}^n$. In Section 4, we first rephrase the four conditions from Section 3 in terms of another matrix, and find a formula for the matrix that depends on the second eigenvalue of the adjacency matrix of the graph. In Section 5, we further reduce the positive-semidefinite conditions, which we use to prove Theorem 1.1 in Section 6.

2 Preliminaries

First, we formally define a 2-distance set and some related concepts.

Definition 2.1. A set of points S in $\mathbb{R}^n$ is a *2-distance set* if

$$|D| = 2 \text{ where } D = \{||p_i - p_j|| \text{ for } p_i, p_j (\neq p_i) \in S\}.$$

Let $D = \{\alpha_1, \alpha_2\}$ such that $\alpha_1 > \alpha_2$. Denote by $k = \alpha_1/\alpha_2 > 1$ the *ratio* of S.

Defining the ratio allows us to consider 2-distance sets only up to scaling. (In this paper, we consider point sets that are the same under isometries and scaling as equivalent sets.)

Definition 2.2. For a 2-distance set S, we define its *associated graph* $G = G(S)$ to be a graph in which every vertex corresponds to a point in S, and two vertices are adjacent if and only if the distance between their corresponding points in S is α_1.

We investigate when a graph has an associated 2-distance set.

Definition 2.3. A graph G is *representable in* $\mathbb{R}^n$ *with ratio* $k > 1$ if there exists a 2-distance set S in $\mathbb{R}^n$ with distance ratio $k > 1$ such that $G = G(S)$ is the associated graph of S.

There are infinitely many non-isomorphic 2-distance sets on at most $n + 1$ points. For instance, we can generate infinitely many 2-distance sets by taking a regular simplex on n points in a hyperplane of dimension $n - 1$ and an arbitrary point on the line through the center of the simplex that is perpendicular to the $(n - 1)$-dimensional hyperplane of the simplex.

The following theorem gives a necessary and sufficient condition for a graph of size $n+2$ to be representble in $\mathbb{R}^n$, thus proving that there are finitely many 2-distance sets with $n+2$ points. It is a direct implication from the proof of Theorem 1 in [4].

Theorem 2.1 (Einhorn and Schoenberg [4])**.** *Let G be a graph on $n + 2$ vertices. If G is a complete multipartite graph, then G is not representable in $\mathbb{R}^n$ for any value of $k > 1$. If G is not a complete multipartite graph, then there exists a unique value $k > 1$ for which G is representable in $\mathbb{R}^n$ with ratio k.*

Theorem 2.1 allows us to work with the graph of the 2-distance set, which is a much simpler structure than the 2-distance set. We also cite the following lemma which was used to prove Theorem 2.1. It gives us a necessary and sufficient condition for the existence of points with a fixed set of distances between them in a Euclidean space of dimension k.

Lemma 2.2 (Schoenberg [7]). *Let* $\{m_{ij}\}_{i,j=1}^{n}$ *be nonnegative real numbers such that* $m_{ij} = m_{ji}$ *for all* i, j *and* $m_{ii} = 0$ *for* $i = 1, 2 \dots n$. *There exist points* $p_1, \dots, p_n \in \mathbb{R}^d$ *with* $||p_i - p_j|| = m_{ij}$ *for all* i, j *if and only if the* $(n-1) \times (n-1)$ *matrix* B' *with*

$$b'_{ij} = m_{1i}^2 + m_{1j}^2 - m_{ij}^2$$

is positive semidefinite (which we denote by $B' \succeq 0$*) and* $\operatorname{rank}(B') \leq d$. *Moreover, this configuration is unique up to congruence.*

Now we give a definition for a spherical 2-distance set, the structure that we analyze throughout this paper. A similar definition is used in the literature [8].

Definition 2.4. A 2-distance set in $\mathbb{R}^n$ is called *spherical* if all of its points lie on an $(n-1)$-dimensional sphere in $\mathbb{R}^n$. A graph G is called *spherical* or we say that G has a *spherical representation* if there exists a spherical 2-distance set S whose associated graph is G.

We denote by A_G the adjacency matrix of a graph $G(V, E)$ and its eigenvalues by $\lambda_1 \geq \lambda_2 \geq \dots \lambda_{|V|}$.

We have now provided all the necessary preliminaries, and we are ready to begin examining conditions for 2-distance sets to have a spherical representation.

3 Conditions for spherical 2-distance sets

In this section we give a set of four necessary and sufficient conditions for a graph to have a spherical representation. First, we define the matrix B_G which encodes the pairwise distances of the corresponding 2-distance set.

Definition 3.1. Given a *representable* graph $G(V, E)$ on $n + 2$ vertices with ratio k, denote by $B_G = B_G(k)$ the $(n+2)\times(n+2)$ matrix with entries

$$b_{ij} = \begin{cases} 0, \text{ if } i = j \\ -1, \text{ if } \{i,j\} \notin E \\ -k^2, \text{ if } \{i,j\} \in E. \end{cases}$$

Next, Proposition 3.1 gives an exact characterization of graphs with $n+2$ vertices that are representable in $\mathbb{R}^n$. Proposition 3.1 is symmetric with respect to all vertices, in contrast with Lemma 2.2, in which one vertex has to be used as a reference. Denote by $\mathbf{1}^\perp$ the orthogonal complement of the span of $\mathbf{1}$ – the all-ones vector.

Proposition 3.1. *Let G be a graph on $n+2$ vertices. Then G is representable in $\mathbb{R}^n$ with ratio k if and only if the following two conditions hold.*

(1) For all vectors $w \in \mathbf{1}^\perp$ we have $w^T B_G(k) w \geq 0$.

(2) There exists a nonzero vector $w \in \mathbf{1}^\perp$ such that $B_G(k)w = \lambda\mathbf{1}$ for a real number λ.

Proof. We define $B'_G = \left(I - \mathbf{1}e_1^T\right) B_G \left(I - e_1\mathbf{1}^T\right)$, which is equivalent to the definition in Lemma 2.2. Thus, G is representable in $\mathbb{R}^n$ if and only if

$$B'_G \succeq 0 \text{ and } \operatorname{rank}(B'_G) \leq n.$$

We show that these two conditions are equivalent to (1) and (2), respectively.

The condition $B'_G \succeq 0$ is equivalent to $v^T B'_G v \geq 0$ for all vectors v. This gives us

$$v^T \left(I - \mathbf{1}e_1^T\right) B_G \left(I - e_1\mathbf{1}^T\right) v \geq 0$$
$$\Leftrightarrow w^T B_G w \geq 0 \text{ where } w = \left(I - e_1\mathbf{1}^T\right) v.$$

Note that $w_1 = -\sum_{i=2}^{n} v_i, w_2 = v_2, w_3 = v_3, \ldots, w_{n+2} = v_{n+2}$, so w is a vector in $\mathbf{1}^\perp$. Conversely, every vector in $\mathbf{1}^\perp$ can be expressed in this form.

The condition $\operatorname{rank}(B'_G) \leq n$ is equivalent to $\dim\operatorname{null}(B_G) \geq 2$. Observe that $e_1 \in \operatorname{null}(B_G)$, so there exists a vector v independent with e_1 such that $\left(I - \mathbf{1}e_1^T\right) B_G \left(I - e_1\mathbf{1}^T\right) v = 0$. This means that there exists $w \perp \mathbf{1}$ such that $\left(I - \mathbf{1}e_1^T\right) B_G w = 0$, so $B_G w \in \operatorname{null}\left(I - \mathbf{1}e_1^T\right) =$

span($\mathbf{1}$), thus there exists a real number λ such that $B_G w = \lambda \mathbf{1}$. Conversely, if there exist a vector satisfying condition (2), then it is in the nullspace of B'_G and is independent of e_1, so $\mathrm{rank}(B'_G) \leq n$. $\square$

We now consider the conditions for a graph to be spherical. Denote by J the all-ones matrix.

Lemma 3.2. *Let G be a graph on $n+2$ vertices that is representable in $\mathbb{R}^n$ with ratio k. It has a spherical representation with ratio k if and only if the following two conditions hold.*

(3) $\det(xJ + B_G)$ *is the zero polynomial.*

(4) There exists $r_0 = r_0(B_G) \in \mathbb{R}$ such that $rJ + B_G \succeq 0$ for all $r \geq r_0$.

Proof. We first show the only if direction. Suppose that G is a graph on $n+2$ vertices which can be representable in $\mathbb{R}^n$ with ratio k, and its corresponding 2-distance set is spherical. Take the center O of the sphere and take a point O_1 on the line in $\mathbb{R}^{n+1}$ through O that is orthogonal to the subspace $\mathbb{R}^n$ where G is representable. The distance between every point from the 2-distance set and O_1 is the same and is denoted by x_1. Note that we can choose any $x_1 \geq \sqrt{r_0/2}$ where r_0 is the radius of the circumsphere of the 2-distance set that corresponds to G. By Lemma 2.2, we see that $B' = 2x_1^2 J + B_G \succeq 0$ for all $x_1 \geq \sqrt{r_0/2}$, so condition (4) holds for $r \geq r_0$, and $\mathrm{rank}(B') \leq n+1$, i.e., $\det(B') = 0$. Since $\det(B')$ is a polynomial in x_1 which vanishes for all $x_1 \geq \sqrt{r_0/2}$, it must be the zero polynomial, so condition (3) holds.

We now show the if direction. Suppose that G is representable in $\mathbb{R}^n$ with ratio k and conditions (3) and (4) hold. Take $r = r_0$. Since $rJ + B_G$ is positive semidefnite and $\mathrm{rank}(rJ + B_G) \leq n+1$, by Lemma 2.2 the 2-distance set corresponding to G is inscribed in a sphere with radius r in $\mathbb{R}^{n+1}$ but it also lies on a subspace isomorphic to $\mathbb{R}^n$, so it lies in the intersection of the sphere and an n-dimensional hyperplane, which is precisely a sphere in $\mathbb{R}^n$. This means that the 2-distance set corresponding to G is spherical. $\square$

By combining Proposition 3.1 and Lemma 3.2, we obtain a set of four conditions that is necessary and sufficient for G to be representable in $\mathbb{R}^n$ with ratio k using only the matrix B_G.

(1) For all vectors $w \in \mathbf{1}^\perp$ we have $w^T B_G w \geq 0$.

(2) There exists a nonzero vector $w \in \mathbf{1}^\perp$ such that $B_G w = \lambda \mathbf{1}$ for a real number λ.

(3) $\det(xJ + B_G)$ is the zero polynomial.

(4) There exists $r_0 = r_0(B_G) \in \mathbb{R}$ such that $rJ + B_G \succeq 0$ for all $r \geq r_0$.

We first simplify condition (3) as follows.

Lemma 3.3. *If a graph G on $n+2$ vertices with a distance ratio k satisfies condition (2), then (3) holds if and only if* $\det(B_G) = 0$.

Proof. If condition (3) holds, then $\det(xJ + B_G) \equiv 0$, as sought.

Now assume that $\det(B_G) = 0$. Thus, there exists a vector $u \neq 0$ such that $B_G u = 0$. From (2), there exists $w \in \mathbf{1}^\perp$ such that $B_G w = \lambda \mathbf{1}$. It therefore suffices to show that for every x, there exists a nonzero vector v such that $(xJ + B_G)v = 0$.

First, if $\lambda = 0$, then $(xJ + B_G)w = 0$ for every x. Otherwise, take $\beta = -\frac{x\langle u,1\rangle}{\lambda}$. Then,

$$\begin{aligned}
(xJ + B_G)(u + \beta w) &= xJu + \beta xJw + B_G u + \beta B_G w \\
&= x\langle u, \mathbf{1}\rangle \mathbf{1} + \beta x \mathbf{0} + \mathbf{0} + \beta\lambda \mathbf{1} \\
&= (x\langle u, \mathbf{1}\rangle + \beta\lambda)\mathbf{1} = 0.
\end{aligned}$$

Note that $u + \beta w \neq 0$ because otherwise u and w would have been collinear, which gives us $Bw = 0$, so $\lambda = 0$. Thus, for every x there exists a nonzero vector v such that $(xJ + B_G)v = 0$, as needed. □

We let $\det(B_G) = 0$ be condition (3′).

4 Second eigenvalue of the adjacency matrix

From Lemma 3.3, we can find a formula for B_G which does not depend on the ratio of the 2-distance set. Conversely, if this formula for B_G holds, then we prove that condition (1) implies conditions (2) and (3′).

First, we transform the matrix B_G to a new matrix $\overline{B_G}$ which is more closely related to the adjacency matrix A_G and rewrite conditions (1)-(4) in terms of $\overline{B_G}$.

Definition 4.1. Denote by $\overline{B_G}$ the matrix $\frac{1}{k^2-1}(B_G + J)$ that has the form $\frac{1}{k^2-1}I - A_G$.

Conditions (1) and (2) for the matrix $\overline{B_G}$ become the following.

$\overline{(1)}$ For all $w \in \mathbf{1}^\perp$ we have $0 \le w^T \overline{B_G} w$.

$\overline{(2)}$ There exists a nonzero vector $w \in \mathbf{1}^\perp$ such that $\overline{B_G} w = \lambda \mathbf{1}$ for some real number λ.

These are clearly equivalent to the original conditions (1) and (2) because $\overline{B_G} w = \frac{1}{k^2-1}(B_G + J)w = \frac{1}{k^2-1} B_G w$ for all $w \in \mathbf{1}^\perp$.

Condition (3) is transformed to

$\overline{(3)}$ $\det(xJ + \overline{B_G}) \equiv 0$,

but this is equivalent to the original condition (3) by a change of variable:

$$\det\left(xJ + \frac{1}{k^2-1}(B_G + J)\right) = \left(\frac{1}{k^2-1}\right)^{n+2} \det\left((x(k^2-1)+1)J + B_G\right).$$

By the same proof as in Lemma 3.3 we see that this is also equivalent to condition (3′) for $\overline{B_G}$, i.e. $\det(\overline{B_G}) = 0$.

Condition (4) becomes the following:

$\overline{(4)}$ There exists $r_0 \in \mathbb{R}$ such that for all $r \ge r_0$ we have $rJ + \overline{B_G} \succeq 0$.

This is equivalent to $B_G + (r(k^2-1)+1)J \succeq 0$ for sufficiently large r, which is the same as the original condition (4) (for B_G).

We now find a formula for $\overline{B_G}$ in terms of the second eigenvalue of the adjacency matrix. This also gives us a formula for B_G that does not depend on k.

Lemma 4.1. *Let G be a graph on $n+2$ vertices. Suppose that it satisfies conditions $\overline{(2)}$, and $\overline{(3)}$ for some $k > 1$, then $B_G = I - \frac{1}{\lambda_2}A_G - J$.*

Proof. Let $m = \frac{1}{k^2-1}$, so $\overline{B_G} = mI - A_G$. By Lemma 3.3, $\det(\overline{B_G}) = 0$, so there exists an eigenvector v of A_G that has eigenvalue $m = \lambda_i$ for some i. By condition $\overline{(1)}$, $x^T\overline{B_G}x \geq 0$ for $x \in \mathbf{1}^\perp$ which is equivalent to $x^T A_G x \leq m x^T x$, i.e., $\frac{x^T A_G x}{x^T x} \leq m$ for every $x \in \mathbf{1}^\perp$. So, by the Courant–Fischer–Weyl min-max [9] principle we have

$$\lambda_2 = \min_{\substack{U \\ \dim(U)=n-1}} \left\{ \max_{x \in U} \frac{x^T A_G x}{x^T x} \right\} \leq \max_{x \in \mathbf{1}^\perp} \frac{x^T A_G x}{x^T x} \leq m.$$

By condition $\overline{(2)}$, there exists a vector $x_1 \in \mathbf{1}^\perp$ such that $x_1^T(mI - A_G)x_1 = 0$ or equivalently $m = \frac{x_1^T A_G x_1}{x_1^T x_1}$. But $m = \lambda_i$ for some i, so $m \leq \lambda_1$.

If $\lambda_1 = \lambda_2$, then $m = \frac{x_1^T A_G x_1}{x_1^T x_1} \leq \lambda_1 = \lambda_2$ by the min-max principle.

If $\lambda_1 > \lambda_2$, then $v_1 \notin \mathbf{1}^\perp$ because λ_1's eigenvector v_1 has nonnegative coordinates. Since $\frac{x^T A_G x}{x^T x} = \lambda_1$ holds if and only if $x \in \operatorname{span}(v_1)$ and $x_1 \notin \operatorname{span}(v_1)$, so $m < \lambda_1$ thus $m \leq \lambda_2$.

Therefore $\lambda_2 \leq m \leq \lambda_2$, so $m = \lambda_2$. Undoing the transformations of B_G, we obtain $B_G = I - \frac{1}{\lambda_2}A_G - J$, as needed. □

This gives us a formula for k in terms of λ_2.

Corollary 4.2. *Given graph G on $n+2$ vertices. If G is spherical, then the distance ratio $k = \sqrt{\frac{1}{\lambda_2} + 1}$. (In particular, this implies that $\lambda_2 > 0$.)*

Proof. From the proof of Lemma 4.1, we have $\frac{1}{k^2-1} = m = \lambda_2$, so the distance ratio $k = \sqrt{\frac{1}{\lambda_2} + 1}$. □

Using Lemma 4.1 we can prove that one of the eigenvectors of λ_2 is in $\mathbf{1}^\perp$.

Lemma 4.3. *Let G be a graph on $n+2$ with ratio k that satisfies conditions $\overline{(1)}$, $\overline{(2)}$, and $\overline{(3)}$. There exists an eigenvector v_2 of λ_2 that is orthogonal to $\mathbf{1}$.*

Proof. Let v_1 be an eigenvector of λ_1 and v_2 be an eigenvector of λ_2 such that $\langle v_1, v_2\rangle = 0$ and let $||v_1|| = ||v_2|| = 1$. The subspace $\text{span}(v_1, v_2)$ has dimension 2, so it intersects $\mathbf{1}^\perp$ in a line or a plane. This means that there exist real numbers α and β such that $\alpha v_1 + \beta v_2 \in \mathbf{1}^\perp$ (where α and β are not both equal to 0). The matrix $\overline{B_G} = \lambda_2 I - A_G$ by Lemma 4.1, so it has v_2 in its nullspace. By condition $\overline{(1)}$,

$$\begin{aligned}
0 &\leq (\alpha v_1 + \beta v_2)^T \overline{B_G}(\alpha v_1 + \beta v_2)\\
&= \alpha^2 v_1^T \overline{B_G} v_1 + \beta^2 v_2^T \overline{B_G} v_2 + \alpha\beta(v_1^T \overline{B_G} v_2 + v_2^T \overline{B_G} v_1)\\
&= \alpha^2(\lambda_2 v_1^T v_1 - v_1^T A_G v_1) + \beta^2 v_2^T \overline{B_G} v_2 + \alpha\beta v_1^T \overline{B_G} v_2 + \alpha\beta(v_1^T \overline{B_G} v_2)^T\\
&= \alpha^2(\lambda_2 v_1^T v_1 - v_1^T A_G v_1)\\
&= \alpha^2(\lambda_2 - \lambda_1).
\end{aligned}$$

We have $\alpha^2(\lambda_2 - \lambda_1) \leq 0$. Since $\alpha^2(\lambda_2 - \lambda_1) \geq 0$ by condition $\overline{(1)}$, which means that either $\lambda_1 = \lambda_2$ or $\alpha = 0$. In the first case λ_2's eigenspace has dimension at least 2, so there is a nonzero intersection with $\mathbf{1}^\perp$, so we ca set $v_2 \in \mathbf{1}^\perp$. In the second case we have $\beta v_2 \in \mathbf{1}^\perp$, and $\beta \neq 0$, so $v_2 \in \mathbf{1}^\perp$. □

From the proof of Lemma 4.3, we can see that the eigenspace of λ_2 is in $\mathbf{1}^\perp$, unless $\lambda_1 = \lambda_2$ in which case we can choose the eigenvectors of λ_2, so that only the eigenvector of λ_1 is not in $\mathbf{1}^\perp$.

Corollary 4.4. *If G is a graph with ratio k that satisfies conditions $\overline{(1)}$, $\overline{(2)}$, and $\overline{(3)}$ then if $\lambda_1 > \lambda_2$, all eigenvectors of λ_2 are in $\mathbf{1}^\perp$, and if $\lambda_1 = \lambda_2$, then all eigenvectors but one can be chosen to be in $\mathbf{1}^\perp$.*

Now, we are ready to prove that if condition $\overline{(1)}$ holds for $\overline{B_G} = \lambda_2 I - A_G$, then conditions $\overline{(2)}$ and $\overline{(3)}$ also hold.

Lemma 4.5. *Given a graph G on $n+2$ vertices and $\overline{B_G} = \lambda_2 I - A_G$. If for every $w \in \mathbf{1}^\perp$ $w^T \overline{B_G} w \geq 0$, then conditions $\overline{(2)}$ and $(3')$ are satisfied.*

Proof. By the proof of Lemma 4.3, if condition $\overline{(1)}$ is true for the matrix $\overline{B_G} = \lambda_2 I - A_G$, then the eigenvector v_2 of the second largest eigenvalue is in $\mathbf{1}^\perp$. From this we can deduce that $\overline{B_G} v_2 = 0$, which implies both conditions $\overline{(2)}$ and $(3')$. $\square$

5 Reduction of the positive-semidefinite conditions

In this section we simplify condition $\overline{(4)}$. Lemma 5.2 reduces condition $\overline{(4)}$ to condition $\overline{(1)}$ and a much weaker condition $\overline{(5)}$. To do this, we use condition $\overline{(1)}$ to reduce condition$\overline{(4)}$ to the existence of an upper bound on $\dfrac{(\mathbf{1}^T A_G v)^2}{\lambda_2 v^T v - v^T A_G v}$ for $v \in \mathbf{1}^\perp$. Lemma 5.1 gives a necessary and sufficient condition for the existence of this bound.

Lemma 5.1. *Let G be a graph on $n+2$ vertices such that $\overline{B_G} = \lambda_2 I - A_G$, and $\overline{B_G}$ satisfies condition $\overline{(1)}$, and let V be a subspace of $\mathbf{1}^\perp$. There exists a real number r such that*

$$r(n+2)(\lambda_2 v^T v - v^T A_G v) \geq (\mathbf{1}^T A_G v)^2 \text{ for every } v \in V \tag{1}$$

if and only if $\mathbf{1}^T A_G v = 0$ for all vectors $v \in V$ such that $\lambda_2 v^T v = v^T A_G v$.

Proof. The forward direction is straightforward.

For the reverse direction, we induct on the dimension of V. The base case $\dim(V) = 0$ is trivial.

For the inductive step, assume that V has dimension at least one and that the statement hold for all subspaces of $\mathbf{1}^\perp$ with lower dimension. Divide both sides of inequality (1) by $v^T v$ to obtain

$$r(n+2)\left(\lambda_2 - \frac{v^T A_G v}{v^T v}\right) \geq \frac{(\mathbf{1}^T A_G v)^2}{v^T v}.$$

By Cauchy-Schwartz's inequality, we can bound the absolute value of the right-hand side $\left|\frac{(\mathbf{1}^T A_G v)^2}{v^T v}\right| \leq ||\mathbf{1}^T A_G||^2$. Denote by

$$f(v) = \left(\lambda_2 - \frac{v^T A_G v}{v^T v}\right).$$

Let $L = \inf_{\substack{v \in V \\ v \neq 0}} f(v)$. Note that $L \geq 0$ by condition $\overline{(1)}$. If $L > 0$, the inequality is true for every $r \geq \frac{||1^T A_G||^2}{(n+2)L}$. If $L = 0$, then as f is invariant under scaling, so L is also the infimum of f on the unit sphere, which is compact. Therefore, this infimum is achieved, i.e., there exists a vector w such that $f(w) = 0$. By the inductive hypothesis the inequality holds for all $v \in V \cap \operatorname{span}(w)^\perp$. If v is not perpendicular to w, by scaling v we can write $v = w + \varepsilon x$ where $||x|| = 1$ and $x \perp w$. The inequality becomes

$$r(n+2)(\lambda_2 + \lambda_2\varepsilon^2 - w^T A_G w - 2\varepsilon x^T A_G w - \varepsilon^2 x^T A_G x) \geq (\mathbf{1}^T A_G w + \varepsilon \mathbf{1}^T A_G x)^2$$

$$r(n+2)(\lambda_2\varepsilon^2 - 2\varepsilon x^T A_G w - \varepsilon^2 x^T A_G x) \geq (\varepsilon \mathbf{1}^T A_G x)^2. \tag{2}$$

Inequality (2) is a quadratic inequality in ε with zero constant term, so the linear term should also be zero, i.e., $x^T A_G w = 0$. Going back to inequality (2) and dividing by ε^2 we get

$$r(n+2)(\lambda_2 - x^T A_G x) \geq (\mathbf{1}^T A_G x)^2$$

for $x \in V \cap \operatorname{span}(w)^\perp$ that has lower dimension than V, so the inductive step is completed. □

We are now equipped to simplify condition $\overline{(4)}$. We prove that it is equivalent to condition $\overline{(1)}$ combined with a new condition $\overline{(5)}$.

Lemma 5.2. *A graph G on $n+2$ vertices is spherical if and only if $w^T \overline{B_G} w \geq 0$ for all vectors $w \in \mathbf{1}^\perp$ where $\overline{B_G} = \lambda_2 I - A_G$, and*

$\overline{(5)}$ $\mathbf{1}^T A_G w = 0$ *for all vectors $w \in \mathbf{1}^\perp$ such that $w^T \overline{B_G} w = 0$.*

Proof. We can easily see that condition $\overline{(4)}$ implies condition $\overline{(1)}$, so if we know that $\overline{(4)}$ holds for the matrix $\overline{B_G} = \lambda_2 I - A_G$, then by Lemma 4.5 we also have conditions $\overline{(2)}$ and $\overline{(3)}$, so G has a spherical representation.

Thus, we need to check that condition $\overline{(1)}$ combined with the fact that $\mathbf{1}^T A_G w = 0$ for all vectors $w \in \mathbf{1}^\perp$ such that $\lambda_2 w^T w = w^T A_G w$ is equivalent to condition $\overline{(4)}$.

Every vector u can be expressed as $\alpha\mathbf{1} + \beta v$ where $v \in \mathbf{1}^\perp$. So, condition $\overline{(4)}$ states that for all α, β, v and sufficiently large r

$$\begin{aligned} 0 &\leq (\alpha\mathbf{1} + \beta v)^T (rJ + \lambda_2 I - A_G)(\alpha\mathbf{1} + \beta v) \\ &= r(\alpha\mathbf{1} + \beta v)^T J(\alpha\mathbf{1} + \beta v) + (\alpha\mathbf{1} + \beta v)^T(\lambda_2 I - A_G)(\alpha\mathbf{1} + \beta v). \end{aligned}$$

As $\operatorname{span}(J) = \operatorname{span}(\mathbf{1})$, the right-hand side expression is equal to

$$\begin{aligned}
&\alpha^2 r(n+2) + (\alpha\mathbf{1} + \beta v)^T(\lambda_2 I - A_G)(\alpha\mathbf{1} + \beta v) \\
&= \alpha^2 r(n+2) + \lambda_2(\alpha\mathbf{1} + \beta v)^T(\alpha\mathbf{1} + \beta v) - (\alpha\mathbf{1} + \beta v)^T A_G(\alpha\mathbf{1} + \beta v) \\
&= \alpha^2 r(n+2) + \lambda_2\alpha^2(n+2) + \lambda_2\beta^2 v^T v - (\alpha\mathbf{1} + \beta v)^T A_G(\alpha\mathbf{1} + \beta v) \\
&= \alpha^2 r(n+2) + \lambda_2\alpha^2(n+2) + \lambda_2\beta^2 v^T v - \alpha^2\mathbf{1}^T A_G\mathbf{1} - \beta^2 v^T A_G v - 2\alpha\beta\mathbf{1}^T A_G v \\
&= \alpha^2\left(\lambda_2(n+2) - \mathbf{1}^T A_G\mathbf{1} + r(n+2)\right) - 2\alpha\beta\mathbf{1}^T A_G v + \beta^2(\lambda_2 v^T v - v^T A_G v).
\end{aligned}$$

If $\alpha = 0$ this becomes condition $\overline{(1)}$.

Assume that $\alpha \neq 0$. The first two terms in the coefficient of α^2 are constants, so we can ignore them and increase r by a constant. So, we want

$$\alpha^2 r(n+2) - 2\alpha\beta\mathbf{1}^T A_G v + \beta^2(\lambda_2 v^T v - v^T A_G v) \geq 0.$$

Thus the discriminant of the quadratic polynomial should be non-positive. That is

$$r(n+2)(\lambda_2 v^T v - v^T A_G v) \geq (\mathbf{1}^T A_G v)^2$$

for sufficiently large r, which reduces the lemma to Lemma 5.1. □

By condition $\overline{(5)}$, for all $w \in \mathbf{1}^\perp$ such that $w^T\overline{B_G}w = 0$ we have $A_G w \in \mathbf{1}^\perp$. It turns out that if conditions $\overline{(1)}$ and $\overline{(5)}$ hold simultaneously, then a stronger condition is true $A_G w = \lambda_2 w$ for all $w \in \mathbf{1}^\perp$ such that $w^T\overline{B_G}w = 0$. Therefore all vectors for which equality holds in condition $\overline{(1)}$ are eigenvectors of A_G with eigenvalue λ_2, i.e., null vectors of $\overline{B_G}$.

Lemma 5.3. *Let G be a graph on $n+2$ vertices. Then G has a spherical representation if and only if condition $\overline{(1)}$ holds for $\overline{B_G} = \lambda_2 I - A_G$, and for every vector $w \in \mathbf{1}^\perp$ such that $w^T\overline{B_G}w = 0$ we have $\overline{B_G}w = 0$.*

Proof. By Lemma 5.2, it suffices to show that if $w^T\overline{B_G}w = 0$, then $A_G w \in \mathbf{1}^\perp$ is equivalent to $\overline{B_G}w = 0$. From the proof of Lemma 5.1 we know that $x^T A_G w = 0$ for all $x \in \mathbf{1}^\perp$ and $x \perp w$ an thus $A_G w \in \operatorname{span}(1, w)$. Therefore $A_G w \in \operatorname{span}(w)$, so w is an eigenvector of A_G. If its eigenvalue is λ, then

$$0 = \lambda_2 w^T w - w^T A_G w = (\lambda_2 - \lambda)w^T w.$$

This means that $\lambda = \lambda_2$, so $\overline{B_G}w = \lambda_2 w - A_G w = 0$. Thus if equality holds in condition $\overline{(1)}$, then $\overline{B_G}w = 0$. On the other hand, if $\overline{B_G}w = 0$, then $\lambda_2 w - A_G w = 0$, so $1^T A_G w = \lambda_2 1^T w = 0$ which is equivalent to $w^T \overline{B_G} w = 0$ by Lemma 5.2. $\square$

6 Proof of the main theorem

We now use Lemma 5.3 to prove our main theorem.

Proof of Theorem 1.1. Let the eigenvalues of PA_GP be $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n+1}$ and 0 where 0 corresponds to the eigenvector $\mathbf{1}$. Suppose that the maximum eigenvalue of PA_GP is λ_2. The matrices PA_GP and $P\overline{B_G}P$ have the same set of eigenvectors. Therefore the spectrum of $P\overline{B_G}P$ is the set $\lambda_2 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_2 - \mu_{n+1}, 0$. Thus all eigenvalues of $P\overline{B_G}P$ are nonnegative, so condition $\overline{(1)}$ holds for $\overline{B_G}$. Conversely, if condition $\overline{(1)}$ holds, then $\mu_1 \geq \lambda_2$. Furthermore, by Lemma 4.5, conditions $\overline{(2)}$ and $\overline{(3)}$ hold. So, by Lemma 4.3, there is an eigenvector with an eigenvalue λ_2 of A_G such that $v_2 \in \mathbf{1}^\perp$, thus v_2 is an eigenvector of PA_GP and $P\overline{B_G}P$, so $\mu_1 \geq \lambda_2$. Thus $\mu_1 = \lambda_2$.

Now, by Lemma 5.3, we have to prove that if we assume condition $\overline{(1)}$ holds for $\overline{B_G}$, then the following two conditions are equivalent.

(a) The multiplicity of λ_2 in A_G (excluding λ_1 if $\lambda_1 = \lambda_2$) is equal the multiplicity of λ_2 in PA_G.

(b) We have $w^T \overline{B_G} w = 0$ if and only if $\overline{B_G}w = 0$ for all $w \in \mathbf{1}^\perp$.

Condition $\overline{(1)}$ holds for $\overline{B_G}$, so by Lemma 4.5, conditions $\overline{(2)}$ and $\overline{(3)}$ hold, thus Corollary 4.4 holds and every eigenvector of λ_2 (except the eigenvector of λ_1 when $\lambda_1 = \lambda_2$) is also an eigenvector of PA_GP, thus the multiplicity of λ_2 in A_G (excluding λ_1 when $\lambda_1 = \lambda_2$) is at most the multiplicity of λ_2 in PA_GP. By condition $\overline{(1)}$ $w^T \overline{B_G} w = 0$ if and only if w is an null vector of $P\overline{B_G}P$, i.e., an eigenvector of λ_2 in A_G. So, if $w^T \overline{B_G} w = 0$ is true only if $\overline{B_G}w = 0$, i.e., w is an eigenvector of λ_2 in A_G, then the the multiplicity of λ_2 in A_G (excluding λ_1 when $\lambda_1 = \lambda_2$) is at least the multiplicity of λ_2 in PA_GP, so they are equal. The reverse also holds. $\square$

We also give an equivalent formulation of Theorem 1.1 using the Cauchy interlacing theorem, which gives a more intuitive understanding of what Theorem 1.1 means.

Theorem 6.1 (Cauchy interlacing theorem [10])**.** *Let A be a symmetric $n \times n$ matrix and B be an $m \times m$ matrix with $B = PAP^*$ where P is an orthogonal projection onto a subspace of dimension m. Then if the eigenvalues of A are $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and the eigenvalues of B are $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_m$, then for all $j \leq m$*

$$\alpha_j \geq \beta_j \geq \alpha_{j+n-m}.$$

In our case, if the eigenvalues of PA_GP are $\mu_1 \geq \mu_2 \geq \dots \mu_{n-1}$, then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_k \geq \mu_k \geq \lambda_{k+1} \geq \cdots \geq \lambda_n.$$

Theorem 1.1 states that $\mu_1 = \lambda_2$ and $\mu_k < \lambda_k = \lambda_2$ where $\lambda_{k+1} < \lambda_k = \lambda_2$.

If G is a regular graph, then the eigenvector of the first eigenvalue is $\mathbf{1}$, thus $\mu_i = \lambda_{i+1}$ for $i = 1, 2 \dots, n-1$ which clearly satisfies Theorem 1.1. We state the following remark.

Corollary 6.2. *If G is a regular graph on $n+2$ vertices that is not complete multipartite, then G has a spherical representation in $\mathbb{R}^n$.*

7 Future developments

We would like to further investigate if other graphs, similarly to regular graphs, satisfy Theorem 1.1. We plan to work on fining an asymptotic formula for the density of the spherical graphs compared to all representable graphs. Continuing our work on characterizing "small" 2-distance sets, the next logical steps are characterizing the 2-distance sets with $n+3$ points.

We could also analyze 2-distance sets with other imposed conditions, such as radial and rotational symmetry.

8 Practical Takeaways

Characterizing the "small" spherical 2-distance sets helps us understand the general structure of the 2-distance sets, and in a broader context k-distance sets. Using Theorem 1.1, we can provide a fast algorithm with complexity of $\Theta(n^{2.3729})$ (i.e. the complexity of the most efficient algorithm for matrix multiplication at the moment) for testing if a graph has a spherical representation. The algorithm can be used to enumerate spherical 2-distance sets. We managed to enumerate all spherical 2-distance sets with at most seven points. The density of the spherical 2-distance sets compared to all 2-distance sets for dimensions 2,3,4,5 are $2/6, 7/27, 42/145, 188/1029$, respectively.

Dimension n	2	3	4	5
Number of spherical 2-distance sets	2	7	42	188
Number of 2-distance sets	6	27	145	1029
Fraction of 2-distance sets that are spherical	0.33	0.26	0.29	0.18

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