The Stable Picard groups of the exterior algebras $E(n)$

Hanming Ye

Under the direction of

David Jongwoon Lee
Massachusetts Institute of Technology
Department of Mathematics

Research Science Institute
August 1, 2023

Abstract

Let $E(n)$ denote the exterior algebra on $n+1$ generators. It is a subalgebra of the dual Steenrod algebra. We are motivated to study modules over this algebra because they provide the E_2 page for the Adams spectral sequence, which is used to compute stable homotopy groups. In 1976, Adams and Priddy introduced the *stable Picard group* of modules over $E(n)$, denoted $\mathrm{StPic}(E(n))$. They conjectured that $\mathrm{StPic}(E(n)) \cong \mathbb{Z} \oplus \mathbb{Z}$ for all n. We prove that this conjecture can be reduced to checking only the case $\mathrm{StPic}(E(2)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Summary

A classic problem in algebraic topology is finding all possible ways to continuously deform a shape X into another shape Y. This proves to be a very difficult task in general. One approach is to compute algebraic approximations of these deformations, called modules. It is conjectured that all modules of a particular type can be factored into the product between two basis modules. We reduce the conjecture from an infinite number of cases to just one.

1 Introduction

A common theme in algebraic topology is to find all the possible continuous deformations from a shape X to another shape Y. These maps are known as homotopies. An example of homotopies from one circle to another is shown in Figure 1

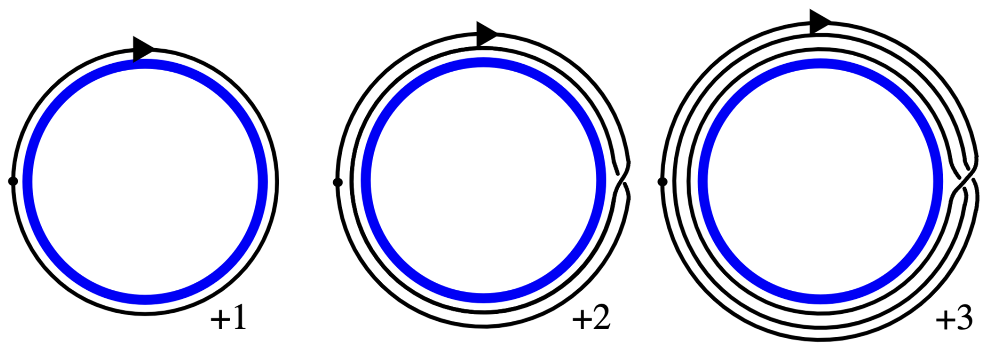

Figure 1: Three possible homotopies from a black circle to a blue circle, labeled by the number of windings present [1]

However, computing homotopies for higher dimensional spaces is extremely difficult, seen through the existence of highly nontrivial examples such as the Hopf map, a homotopy from a 3-sphere to a 2-sphere.

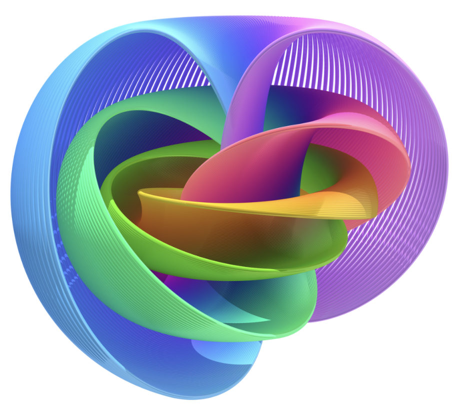

Figure 2: A stereographic projection of the Hopf map [2]

Modern approaches study these topological concepts by converting them to algebraic structures. In 1958, J.F. Adams [3] found that groups of homotopies can be computed from modules M over certain algebras, including the *Steenrod algebra* A_p, the *Hopf algebra* $\mathcal{A}_p(n)$, and the *exterior algebra* $E_p(n)$. Intuitively, modules over algebras may be thought of as generalizations of vector spaces where the scalar product is not necessarily invertible. Here, the parameter n specifies the number of generators of the algebra, similar to a measure of dimension. The modules M form a group under multiplication called the *stable Picard group*, denoted $\mathrm{StPic}(A_p), \mathrm{StPic}(\mathcal{A}_p(n))$, or $\mathrm{StPic}(E_p(n))$ depending on the underlying algebra.

This connection between homotopies and StPic motivates research on the exact structure of StPic. In 1976, Adams and Priddy [4] proved that $\mathrm{StPic}(E_2(1))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. In other words, the modules over the algebra $E_2(1)$ behave like pairs of integers. The same paper proved that $\mathrm{StPic}(A_2(1)) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$, where the $\mathbb{Z}/2$ component is generated by an exotic module known as the Joker. However, in 2017 Bhattacharya and Ricka [5] proved that $\mathrm{StPic}(A_2(2)) \cong \mathbb{Z} \oplus \mathbb{Z}$, with Pan and Yan [6] later proving that $\mathrm{StPic}(A_2(n)) \cong \mathbb{Z} \oplus \mathbb{Z}$ for all $n \geq 2$. This shows that no exotic element exists over larger Hopf algebras. A similar isomorphism is expected to be true for modules over $E_p(n)$: Adams and Priddy conjectured that $\mathrm{StPic}(E_p(n)) \cong \mathbb{Z} \oplus \mathbb{Z}$ for all natural numbers n and any prime p. We show that the conjectured isomorphism for any n depends only on the verification of the case $n = 2$, as in Theorem 1.1.

Theorem 1.1. *Let p be a prime. If $StPic(E_p(2)) \cong \mathbb{Z} \oplus \mathbb{Z}$, then $StPic(E_p(n)) \cong \mathbb{Z} \oplus \mathbb{Z}$ for all integers $n \geq 1$.*

The paper is structured as follows. In Section 2, we define a stable Picard group more rigourously, recall results from literature that we use in the paper, and establish the map $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{StPic}(E_p(n)))$ which we show is an isomorphism. In Section 3, we prove that the map φ is injective and introduce the base case necessary for induction. In Section 4, we show that restrictions of M to a subalgebra of $E_p(n)$ do not affect its structure in StPic, thereby strengthening our inductive hypothesis. In Section 5, we introduce the notion of the *annihilator* of an element $x \in M$ and provide results characterising its behavior. Finally, in Section 6, we prove the main Theorem 1.1 by analysing the annihilator of an element in M.

2 Preliminaries

In 1958, J.F. Adams [3] showed that the cohomology of the mod p *Steenrod algebra* A_p can be used to compute the p-components of the stable homotopy groups of spheres. Furthermore, these cohomology groups can be considered as modules over A_p. Later, Milnor [7] showed that the dual of the Steenrod algebra can be expressed as a tensor product between a polynomial algebra and the graded *exterior algebra* E_p over the field $k = \mathbb{F}_p$ generated by the unit 1 and the symbols Q_i of degree $2p^i - 1$, where $i \geq 0$.

Let N be a finite subset of $\{Q_0, Q_1, \ldots\}$. Then, let $E(N)$ denote E_p restricted to only the generators 1 and $Q_i \in N$. When $N = \{Q_0, \ldots, Q_n\}$ for an integer n, we also write $E(N) = E(n)$, as to agree with conventional notation in literature. We also drop the subscript p, as we maintain the assumption that the base field is $k = \mathbb{F}_p$ throughout the paper. Recall that $Q_i^2 = 0$ and $Q_iQ_j = -Q_jQ_i$ for any two generators Q_i and Q_j by definition of an exterior algebra, where concatenation denotes the wedge product. We investigate the structure of the *stable Picard group* of modules over $E(N)$ and $E(n)$.

2.1 Stable Picard Group

Adams and Priddy showed that the usual topological concepts in stable homotopy theory can be translated into algebraic definitions, which we now provide.

Definition 2.1 (Stable equivalence)**.** Two $E(N)$-modules A and B are *stably equivalent* (denoted $A \simeq B$) if there exist free modules F_1 and F_2 over $E(N)$ such that $A \oplus F_1 \cong B \oplus F_2$. In the sequel we write *equivalence* to stand for stable equivalence.

Definition 2.2 (Stable invertibility)**.** An $E(N)$-module A is *invertible* if there exists another $E(N)$-module B such that $A \otimes B \simeq k$, where $\otimes$ denotes the tensor product. Recall that k denotes the base field of the underlying algebra $E(N)$.

Under the operation of tensor products, the set of invertible $E(N)$-modules forms a group up to equivalence, called the *stable Picard group of* $E(N)$ and denoted by $\mathrm{StPic}(E(N))$. The identity of this group is the base field k (considered as module over $E(N)$).

Adams and Priddy [4] showed that each module M in $\mathrm{StPic}(E(1))$ can be uniquely factored into the product of two invertible basis modules I and $k[1]$. That is, $M = I^a \otimes (k[1])^b$ for some integers a and b. We provide our definition and notation for the invertible modules I and $k[1]$

Definition 2.3 (Desuspension I)**.** Over the algebra $E(N)$, the *desuspension* I is defined as the module generated by all $Q_i \in N$.

Example 2.1. Over $E(n)$, $I = \langle Q_0, Q_1, \ldots, Q_n \rangle$.

Definition 2.4 (Grading shift $k[1]$)**.** Let $k[1]$ denote the module consisting of the base field k in degree 1 and 0 elsewhere. For any integer a, $k[1]^a = k[a]$, which consists of k in degree a and 0 elsewhere. We call this a *grading shift of k by a*.

Adams and Priddy [4] also showed that $(I^a \otimes k[b]) \otimes (I^c k[d]) = I^{a+c} \otimes k[b+d]$. This implies that the map $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{StPic}(E(1))$ given by $(a, b) \to I^a \otimes k[b]$ is a homomorphism [4]. It is this map that we later extend to $\mathrm{StPic}(E(N))$ and prove to be an isomorphism. In the rest of the paper, we denote the tensor product with concactenation ($I^a \otimes k[b] = I^a k[b]$).

2.2 Margolis homology groups

One crucial tool used in the investigation of stable Picard groups are *Margolis homology groups*. We define them in the context of our problem and state results that we employ later.

Definition 2.5 (Margolis homology with respect to Q_i)**.** For any module M over $E(N)$, we can consider each generator Q_i of $E(N)$ as a differential on the chain complex

$$M \xrightarrow{Q_i} M \xrightarrow{Q_i} M.$$

Because $Q_i^2 = 0$ by definition of an exterior algebra, $\mathrm{Im}\ Q_i \subseteq \ker Q_i$. Therefore, we can assign to M and Q_i the Margolis homology group

$$H(M; Q_i) := \ker Q_i / \mathrm{Im}\ Q_i.$$

One utility of Margolis homologies is their exact characterization of invertible modules.

Theorem 2.1 (Adams and Priddy [4]). *An $E(N)$ module M is invertible if and only if the Margolis homologies $H(M; Q_i)$ are 1-dimensional over the base field k for all $Q_i \in N$.*

Margolis homologies also give a necessary and sufficient condition for two modules to be equivalent.

Theorem 2.2 (Adams and Margolis [8]). *Let A and B be $E(N)$ modules. Then A and B are equivalent if and only if there exists a homomorphism $f : A \to B$ that induces isomorphisms $H(A; Q_i) \cong H(B; Q_i)$ for all $Q_i \in N$.*

Because homologies are often easier to compute, we use Theorem 2.2 in the proof of Theorem 3.1 and Lemma 5.1.

2.3 Graded structure

Recall that both $E(N)$ and its modules have grading structures. This invariant serves as a crucial tool that we use extensively. Here we state key facts about their graded structure.

Recall that a graded algebra A is an algebra that can be decomposed into a direct sum

$$A = \bigoplus_{i=0}^{\infty} A_i = A_0 \oplus A_1 \oplus \cdots$$

where A_i are additive groups such that $a_i a_j \in A_{i+j}$ for all elements $a_i \in A_i$ and $a_j \in A_j$. A nonzero element $a \in A_i$ is said to be *homogeneous of degree i* and we denote its degree by $|a| = i$. For example, over the algebra $\mathbb{C}[x]$ of polynomials, this notion of degree is exactly the conventional one. Additionally, unless otherwise specified, *homomorphism* refers to a homomorphism that respects the grading of its domain and image. That is, if f is a homomorphism, then for any elements m (in a graded algebra or module) we have $|f(m)| = |m|$.

This grading structure on modules over $E(n)$ is inherited by the Margolis homologies of M (Definition 2.5). In fact, we can determine the exact grading of a Margolis homology for modules generated by a desuspension I and a grading shift of k.

Lemma 2.3 (Adams and Priddy [4]). *If M is equivalent to $I^a k[b]$ for some integers a and b, then $H(M; Q_i)$ consists of the base field k in degree $a|Q_i| + b$ and 0 in all other degrees.*

We have now provided all the necessary preliminaries, and we are ready to establish the results necessary for induction on $|N|$ in $\mathrm{StPic}E(N) \cong \mathbb{Z} \oplus \mathbb{Z}$.

3 Injectivity of φ

To show that φ is an isomorphism, we must show that it is injective and surjective. We prove that it is always injective by solving a system of equations.

Theorem 3.1. *The map $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to StPic(E(N))$ given by $(a, b) \mapsto I^a k[b]$ is injective for any set of generators N with $|N| \geq 2$.*

Proof. Suppose $\varphi((a, b)) = \varphi((c, d))$ for some integers a, b, c, and d. Then $I^a k[b] \simeq I^c k[d]$. Because $|N| \geq 2$, we can pick two generators Q_i and Q_j of $E(N)$. From Theorem 2.2, we see that

$$H(I^a k[b]; Q_i) \cong H(I^c k[d]; Q_i) \quad \text{and} \quad H(I^a k[b]; Q_j) \cong H(I^c k[d]; Q_j).$$

Then Lemma 2.3 implies that

$$a|Q_i| + b = c|Q_i| + d,$$
$$a|Q_j| + b = c|Q_j| + d.$$

However, $|Q_i| \neq |Q_j|$ when $i \neq j$. Thus, the only solution to this linear system is $a = c$ and $b = d$, showing that φ is injective. □

To show that $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{StPic}(E(N))$ is also surjective for all sets N with $|N| \geq 2$, we induct on the cardinality of N. Adams and Priddy [4] proved the case where $N = \{Q_0, Q_1\}$, but did not use the particular degrees of the generators in their argument (i.e. $|Q_i| = 2p^i - 1$), meaning that their proof easily generalizes to all sets N with $|N| = 2$. This establishes the base case $|N| = 2$.

For the inductive step, suppose it has been shown that all invertible modules over $E(S)$ where $|S| = n$ are of the form $I^a k[b]$ for some integers a and b. We aim to show that all invertible modules over $E(N)$ where $|N| = n + 1$ are expressible in this form as well (Theorem 1.1).

4 Stable structure under restrictions to subalgebras

For an $E(N)$-module M, we write $M|_{E(S)}$ as the $E(S)$-module formed by restricting M to $E(S)$. In particular, we define it as follows.

Definition 4.1. If M is a module over $E(N)$, by definition it is an Abelian group G with a binary operation $E(N) \times G \to G$. We denote $M|_{E(S)}$ by the module formed by the same Abelian group G with the binary operation restricted to the domain $E(S) \times G$.

First, we show that the stable structure of any invertible M remains constant under this restriction operation. This allows us to better leverage the inductive hypothesis by considering M as a module over $E(S)$ where $|S| \leq n$.

Lemma 4.1. *Let N be a set of generators Q_i, and let S be any subset of N. If M is an invertible $E(N)$ modules such that $M \simeq I^a k[b]$ as $E(N)$-modules, then $M|_{E(S)} \simeq I^a k[b]|_{E(S)}$ as $E(S)$-modules.*

Proof. If M is stably equivalent to $I^a k[b]$, then (by definition) there exists free modules F_1 and F_2 over $E(N)$ such that

$$M \oplus F_1 \cong I^a k[b] \oplus F_2.$$

Now we show that $F_1|_{E(S)}$ and $F_2|_{E(S)}$ are still free, which implies

$$M|_{E(S)} \oplus F_1|_{E(S)} \cong I^a k[b]|_{E(S)} \oplus F_2|_{E(S)}$$

and thus $M|_{E(S)} \simeq I^a k[b]|_{E(S)}$. By definition of a free module, F_1 has an $E(N)$ basis $B_N = \{b_1, \dots, b_d\}$ for some d. We create a new $E(S)$ basis B_S that spans $F_1|_{E(S)}$. Note that we can start with B_N and then add all the elements of F_1 which are not spanned by this basis under the restriction to $E(S)$. More specifically, the set

$$B_S = \{b_1, \dots, b_d\} \cup \{Q_i b_j \mid Q_i \in N, Q_i, \notin S, 1 \leq j \leq d\}$$

provides an $E(S)$ basis for $F_1|_{E(S)}$. For example, when $N = \{Q_0, \dots Q_5\}$ and $S = \{Q_0, \dots Q_4\}$, this basis is $\{b_1, \dots, b_d, Q_5 b_1, \dots, Q_5 b_d\}$.

We can verify that B_S spans $F_1|_{E(S)}$ because any $E(N)$-linear combination on $b_1, \dots, b_d$ can be expressed as a $E(S)$-linear combination in this basis. Furthermore, the $E(S)$-spans of the elements in B_S have trivial intersections, implying linear independence. Hence, we have found a basis for $F_1|_{E(S)}$, showing that it is free. Replacing F_1 with F_2 in this argument shows that $F_2|_{E(S)}$ is also free and proves the desired statement. □

Lemma 4.1 shows that restrictions of a module do not affect its stable structure. Recall that under the inductive hypothesis, we assume that all invertible modules over $E(S)$ where $|S| = n$ are of the form $I^a k[b]$. Let M be an invertible module over $E(N)$ where $|N| = n+1$. In Theorem 4.2, we prove the stronger result that all restrictions of M are equivalent to $I^a k[b]$, even if we have no information about M itself.

Theorem 4.2. *Let N be a set of generators Q_i such that $|N| \geq 4$ and let S be any proper subset of N. If M is an invertible $E(N)$ module, then $M|_{E(S)} \simeq I^a k[b]|_{E(S)}$ for some fixed pair of intgers a and b.*

In the proof, we create subsets of N with large intersections, examine M restricted to these intersections, and apply Lemma 4.1 to recover information about M itself.

Proof. The restriction of M to $M|_{E(S)}$ does not change the kernel or image of Q_i as long as $Q_i \in S$. Thus, $H(M; Q_i) = H(M|_{E(S)}; Q_i)$ for all such Q_i. Because M is invertible, it follows from Theorem 2.1 that all Margolis homologies $H(M; Q_i)$ and thus all $H(M|_{E(S)}; Q_i)$ are 1-dimensional over k. This shows that any $M|_{E(S)}$ is invertible as well.

First, we show that the desired statement holds when S contains all of N except one generator. Let A and B be two subsets of N with $|A| = |B| = |N| - 1$. Because A and B are invertible, the inductive hypothesis implies that there exist integers a, b, c, d such that

$$M|_{E(A)} \simeq I^a k[b]|_{E(A)} \quad \text{and} \quad M|_{E(B)} \simeq I^c k[d]|_{E(B)}.$$

We proceed to show that $a = c$ and $b = d$. Consider the intersection C of A and B. From Theorem 3.1, the map $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{StPic}(E(C))$ given by $(x, y) \mapsto I^x k[y]$ is injective because $|C| \geq |N| - 2 \geq 2$. Thus, $I^a k[b]|_{E(C)} \simeq I^c k[d]|_{E(C)}$ implies $a = c$ and $b = d$. The pair a, b is precisely the fixed pair of integers in the statement of the lemma.

Now consider any proper subset S of N. Because S is proper, it is a subset of $S' \subset N$ where $|S'| = |N| - 1$. We have just shown that $M|_{S'} \simeq I^a k[b]|_{E(S')}$. From Lemma 4.1, we see that $M|_S \simeq I^a k[b]|_{E(S)}$. This proves the lemma. □

Remark 4.1. Note that this theorem holds only when $|N| \geq 4$ because otherwise the intersection C may not contain at least two generators, the minimum needed for injectivity as guarenteed Theorem 3.1 to hold. In fact, the map $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{StPic}(E(N))$ is *not* injective when $|N| = 1$. This is because there *are* non-trivial solutions to each individual equation in the proof of Theorem 3.1. For example, suppose that $N = \{Q_0\}$. Then

$$I^a k[-a|Q_0|] = k$$

for any integer a. Due to this failure of injectivity, Theorem 1.1 only holds under the assumption that $\mathbb{Z} \oplus \mathbb{Z} \cong \mathrm{StPic}(E(N))$ where $|N| = 3$. We provide partial results on this case later in Section 7.

5 Extremal cases of the Annihilator A_x

Let M be a module over $E(N)$. We define the *annihilator* A_x of an element $x \in M$ as

$$A_x \coloneqq \{Q_i \in N \mid Q_i x = 0\}.$$

In the proof of Theorem 1.1, we show that for some element $x \in M$, the annihilator A_x is either all of N or empty. In this section, we provide lemmas that characterize the two cases. We start by showing that in the case that A_x is the entire set N, we can determine M.

Lemma 5.1. *An invertible $E(N)$-module M is stably equivalent to k if and only if there exists an element $x \in M$ such that $Q_i x = 0$ and $x \notin Im\ Q_i$ for all $i \in N$ (i.e. $A_x = N$).*

Proof. Suppose $M \simeq k$; by Theorem 2.2 there exists a homomorphism $f : M \to N$ that induces the isomophisms $H(M; Q_i) \cong H(N; Q_i)$ for all $Q_i \in N$. Note that setting $x = f(1)$ gives the desired properties because x is a generator of $H(M; Q_i)$ that satisfies

$$Q_i f(1) = f(Q_i) = f(0) = 0.$$

Now we prove the converse. Because M is invertible, its Margolis homologies $H(M; Q_i)$ are 1-dimensional over k by Theorem 2.1. If an element $x \in M$ satisfies $Q_i x = 0$ for all Q_i, then x generates all $H(M; Q_i)$. Because isomorphisms between 1-dimensional spaces are determined by mapping one generator to another, the map $f : 1 \mapsto x$ induces isomorphisms on the Margolis homologies of k and M, showing that $M \simeq k$ by Theorem 2.2. □

We have characterised the case when $A_x = N$. Next, we consider the case when A_x is empty. That is, $Q_i x \neq 0$ for all $x \in M$.

Remark 5.1. This case presents difficulty because multiplying x by more than one generator could still yield 0. For example, $A_{Q_0+Q_1} = \{\varnothing\}$ but $Q_0 Q_1 (Q_0 + Q_1) = 0$. To analyse these possibilities, we introduce the following notation.

Definition 5.1. Let S be a set of generators Q_i. We write $\min(S)$ for the index of the generator in S with the lowest degree. Likewise, $\max(S)$ denotes the index of the generator with the highest degree. For example, if $S = \{Q_0, Q_1, Q_3\}$, then $Q_{\min(S)} = Q_0$ and $Q_{\max(S)} = Q_3$.

Definition 5.2. Let S be a set of generators Q_i. We define the *product of* S as

$$\Pi S \coloneqq Q_{i_1} Q_{i_2} \cdots Q_{i_{|S|}}$$

where $i_1 = \min(S)$, $i_{|S|} = \max(S)$, and $i_1 < i_2 < \cdots < i_{|S|}$. For example, if $S = \{Q_0, Q_1, Q_3\}$, the product ΠS is $Q_0 Q_1 Q_3$.

In Lemma 5.2, we consider the case where not only $A_x = \{\varnothing\}$ but also the stronger condition of $\Pi N x \neq 0$. We see that x generates a free module that we can discard from M.

Lemma 5.2. *Let N be a set of generators Q_i and M a module over $E(N)$. If $\Pi N x \neq 0$ where $x \in M$, then M contains a free module generated by x. That is $M \cong M' \oplus E(N)$ for some $E(N)$ module M'.*

Proof. Consider the short exact sequence

$$0 \to \langle x \rangle \xrightarrow{a} M \xrightarrow{b} M' \to 0, \tag{1}$$

where $\langle x \rangle$ denotes the free module generated by x under left $E(N)$-action. Here, we consider the differential a as the inclusion map that sends x to x. Then, because the sequence is exact, the kernel of b is the image of a, which is the submodule $\langle x \rangle \subset M$. Thus, M' is equal to the quotient $M/\ker a = M/\langle x \rangle$, the cokernel of a. We show that this M' is exactly the module satisfying the direct sum in the statement of the lemma.

Note that $\langle x \rangle$ is a free module because it has the $E(N)$-basis $\{x\}$. In Adams and Margolis [8], Theorem 4.1 proves that free $E(N)$-modules are also injective. So $\langle x \rangle$ is also injective. In consequence, the short exact sequence (1) splits (see, for example, [9]). That is, $M = M' \oplus \langle x \rangle$. Because $\langle x \rangle$ is isomorphic to $E(N)$, we have $M \cong M' \oplus E(N)$. □

In this section we analysed the annihilator A_x, which allows us to deduce the structure of M by computing products. Together with the results on the restrictions of M in Section 4, we are now equipped to complete the inductive step.

6 Inductive step

Let M be an invertible module over $E(N)$. Recall that we aim to prove that $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{StPic}(E(N))$ is surjective by showing that any M is of the form $I^x k[y]$ for some integers x and y when $|N| = n+1$.

From Theorem 4.2, we know that any restriction of M to a proper subset S of N satisfies $M|_{E(S)} \simeq I^a k[b]|_{E(S)}$ for some fixed pair of integers a and b. If we prove that $I^{-a}k[-b]M$ is of the form $I^x k[y]$, then we can multiply this $E(N)$ module by $I^a k[b]$ to see that $M \simeq I^{x+a}k[y+b]$. Therefore, without loss of generality, we can assume that

$$M|_{E(S)} \simeq k|_{E(S)}$$

by replacing M with $I^{-a}k[-b]M$. We proceed to show that $M \simeq k$ as $E(N)$ modules.

First, Lemma 6.1 below shows that the Margolis homology groups agree with this expected equivalence of M. This result serves as a tool to compare different restrictions of M.

Lemma 6.1. *If $M|_{E(S)} \simeq k|_{E(S)}$, each Margolis homology $H(M; Q_i)$ is k in degree 0 for all $Q_i \in N$.*

Proof. For any Q_i, pick a subset S with at least two elements that contains Q_i. Then $M|_{E(S)} \simeq k|_{E(S)}$, which implies that $H(M; Q_0)$ is k in degree 0 by Lemma 2.3. □

Remark 6.1. Although the Margolis homologies of M are isomorphic to that of k as $E(N)$ modules, Theorem 2.2 does not guarantee that M is stably equivalent to k because this isomorphism is not necessarily induced by some map from M to k.

To show that $M \cong k \oplus F$, the core idea is to analyse how Q_i acts on some element $x \in M$. Then, we can use Lemma 5.1 and Lemma 5.2 to show that x must generate either k or F. To ensure that this element x behaves like a generator, we take it to be the element of minimal degree in M. The proceeding Lemma 6.2 shows that A_x is either all of N or empty.

Lemma 6.2. *Let M be an $E(N)$ module such that $M|_{E(S)} \simeq k|_{E(S)}$ for all proper subsets S of N. If $x \in M$ is of minimal degree, then A_x is either N or $\{\varnothing\}$.*

Proof. Suppose $Q_i x = 0$ for some $Q_i \in N$. We show that this leads to $A_x = N$. Observe that x cannot be in the image of Q_0, as that would imply the existence of $y \in M$ such that $Q_0 y = x$, which would contradict the minimality of $|x|$. Because x is in the kernel but not the image of Q_0, it is a generator of $H(M; Q_0)$. From Lemma 6.1, we see that $H(M; Q_0)$ is k in degree 0, meaning that $|x| = 0$.

Suppose for the sake of contradiction that $Q_j x \neq 0$ for some $Q_j \in N$. Consider $M|_{E(\{Q_0, Q_j\})}$. This module is stably equivalent to k by assumption, so there exists some element $x' \in M|_{E(\{Q_0,Q_j\})}$ such that $Q_0 x' = Q_j x' = 0$ by Lemma 5.1. Because x and x' are both generators of the one-dimensional homology $H(M; Q_0)$, there exists a constant $c \in k$ such that $x = cx'$. Then

$$Q_j x = c Q_j x' = 0,$$

contradicting our assumption that $Q_j x \neq 0$. Thus, $Q_i x = 0$ implies $A_x = N$. The only other possibility is that $Q_i x \neq 0$ for all $Q_i \in N$, which leads to $A_x = \{\varnothing\}$. $\square$

If A_x is all of N, Lemma 5.1 implies that $M \simeq k$, proving the desired statement. So we proceed with the assumption that $A_x = \{\varnothing\}$. Now there are two possibilities for $\Pi N x$: it is either zero or non-zero. If it is non-zero, then Lemma 5.2 implies that there exists an $E(N)$ module M' such that $M = M' \oplus F$ where F is a free module. Because $M \simeq M'$ by definition, we can pick another minimal degree element x' of M' and apply Lemma 6.2 to reach the same casework ($\Pi N x'$ is either zero or nonzero). Because M is finitely generated by assumption, this process of taking out free modules cannot continue indefinitely, and $\Pi N x' = 0$ at some point.

We have discussed the cases $A_x = N$ and$A_x = \{\varnothing\}$ with $\Pi N x \neq 0$. The only possibility left is $A_x = \{\varnothing\}$ with $\Pi N x = 0$ where x is a minimal degree element in M. We show that this is impossible, which would imply $M \simeq k$ and prove Theorem 1.1.

Let $N' = N \setminus \{Q_{\max(N)}\}$. That is, N' is N without its largest generator. Then, we rewrite $\Pi N x = 0$ as

$$Q_{\max(N)}(\Pi N') = 0.$$

Equivalently, $\Pi N' \in \ker Q_{\max(N)}$. In $H(M; Q_{\max(N)})$, the element $\Pi N'$ is either a generator or the zero-class (corresponding to $\Pi N' \in \mathrm{Im}\ Q_{\max(N)}$). We show that both cases lead to contradictions. We do this by creating an inequality on degrees of the generators Q_i.

Theorem 6.3. *Let S be any finite set of generators Q_i. Then*

$$|\Pi S| < |Q_{\max(S)+1}|.$$

That is, the degree of the product of all generators in S is less than the degree of the next highest generator outside S.

Proof. Recall from Section 2 that $|Q_i| = 2p^i - 1$ due to $E(n)$ being subalgebras of the dual Steenrod algebra.

Observe that

$$\begin{aligned}
|\Pi S| &\leq |Q_0 Q_1 \cdots Q_{\max(S)}| \\
&\leq \sum_{i=0}^{\max(S)} 2p^i - 1 \\
&\leq 2 \cdot \frac{p^{\max(S)+1} - 1}{p-1} - \frac{(\max(S)+1)(\max(S)+2)}{2} \\
&\leq \frac{(2p^{\max(S)+1} - 1) - 1}{p-1} \\
&< |Q_{\max(S)+1}|.
\end{aligned}$$

$\square$

We apply Theorem 6.3 to our casework and show that $\Pi N'$ cannot be the zero class in $H(M; Q_{\max(N)})$.

Corollary 6.4. *Let S be any finite set of generators Q_i. If $x \in M$ has minimal degree, then $(\Pi S)x$ is not in the image of Q_j where $j \geq \max(S)$.*

Proof. Any element in the image of Q_j is in the form $Q_j y$ for some $y \in M$. However, by Theorem 6.3,

$$|Q_j y| = |Q_j| + |y| > |(\Pi S)x|$$

because $|Q_j| > |\Pi S|$ and $|y| \geq |x|$. □

Because $\max(N) > \max(N')$ by definition of N', Corollary 6.4 demonstrates that $\Pi N'$ is not in the image of $Q_{\max(N)}$. This forces $\Pi N'$ to generate $H(M; Q_{\max(N)})$. However, Theorem 6.5 shows that this is also impossible.

Theorem 6.5. *If $M|_{E(S)} \simeq k|_{E(S)}$ for all proper subsets S of N. If $x \in M$ is of minimal degree, then there is no generator of $H(M; Q_{\max(N)})$ in the form Sx for any S.*

Proof. From Lemma 6.1, we see that the Margolis homology $H(M; Q_{\max(N)})$ is k in degree 0. However,

$$|Sx| > 0 \text{ because } |S| > 0 \text{ and } |x| = 0.$$

Thus, Sx cannot be a generator if S is non-empty. □

We began by picking a minimal element $x \in M$ where M is an $E(N)$ module. Then, Lemma 6.2 shows that A_x is either N or $\{\varnothing\}$. In the former case, M is equivalent to k. In the latter case, Corollary 6.4 and Theorem 6.5 show that we must have $(\Pi N) \neq 0$. Therefore, we can reduce M to M' and pick another minimal element. Repeating this process a sufficient number of times leads to $A_x = 0$ and $M \simeq k$. Thus, all invertible $E(N)$ modules are of the form $I^a k[b]$.

Together with injectivity proved in Theorem 3.1, we conclude that the map $\varphi : \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{StPic}(E(N))$ given by $(a, b) \mapsto I^a k[b]$ is an isomorphism for all sets N with $|N| \geq 2$. This proves Theorem 1.1.

7 Analysis of $|N| = 3$ and Future Work

As discussed in Remark 4.1, Theorem 1.1 relies on Theorem 4.2, which has only been proven for $|N| \geq 4$. In this section, we extend Theorem 4.2 to several cases of modules M over $E(N)$ where $|N| = 3$, and present possible approaches for future research.

Let $N = \{Q_\alpha, Q_\beta, Q_\gamma\}$ where $\alpha < \beta < \gamma$, and let M be a module over $E(N)$. The base case $|N| = 2$ proven by Adams and Priddy [4] implies that there exist integers a and b such that

$$M|_{E(\{Q_\alpha, Q_\beta\})} \simeq I^a k[b]|_{E(\{Q_\alpha, Q_\beta\})}.$$

Without loss of generality, we can replace M with $I^{-a}k[-b]M$ so that $M|_{E(\{Q_\alpha, Q_\beta\})} \simeq k|_{E(\{Q_\alpha, Q_\beta\})}$. Note that Theorem 4.2 does not imply $M|_{E(S)} \simeq k|_{E(S)}$ for any proper subset

$S \subset N$ because $|N| < 4$. Nevertheless, by comparing Margolis homologies, we characterise the stable structure of $M|_{E(S)}$ in Lemma 7.1.

Lemma 7.1. *If M is a module over $E(\{Q_\alpha, Q_\beta, Q_\gamma\})$ such that $M|_{E(\{Q_\alpha,Q_\beta\})} \simeq k|_{E(\{Q_\alpha,Q_\beta\})}$, then*

$$\begin{aligned} M|_{E(\{Q_\alpha,Q_\gamma\})} &\simeq I^n k[-n|Q_\alpha|]|_{E(\{Q_\alpha,Q_\gamma\})}, \\ M|_{E(\{Q_\beta,Q_\gamma\})} &\simeq I^m k[-m|Q_\beta|]|_{E(\{Q_\beta,Q_\gamma\})}, \\ n(|Q_\gamma| - |Q_\alpha|) &= m(|Q_\gamma| - |Q_\beta|). \end{aligned}$$

for some integers n and m.

In the proof of Lemma 7.1, we form equations characterising $M|_{E(\{Q_\alpha,Q_\gamma\})}$ and $M|_{E(\{Q_\beta,Q_\gamma\})}$ by calculating the degree of each Margolis homology of M in two different ways.

Proof. By the base case $|N| = 2$, we have the equivalences

$$M|E(\{Q_\alpha, Q_\gamma\}) \simeq I^n k[n']|_{E(\{Q_\alpha,Q_\gamma\})} \quad \text{and} \quad M|E(\{Q_\beta, Q_\gamma\}) \simeq I^m k[m']|_{E(\{Q_\beta,Q_\gamma\})}$$

for some integers n, n', m, m'. We proceed to express n' and m' in terms of n and m respectively.

Applying Lemma 2.3 to $M|_{E(\{Q_\alpha,Q_\beta\})} \simeq k|_{E(\{Q_\alpha,Q_\beta\})}$ shows that $H(M; Q_\alpha)$ and $H(M; Q_\beta)$ are both k in degree 0. Applying the same theorem to $M|_{E(\{Q_\alpha,Q_\gamma\})}$ shows that

$$n|Q_\alpha| + n' = |H(M; Q_\alpha)| = 0 \implies n' = -n|Q_\alpha|.$$

Similarly,

$$m|Q_\beta| + m' = |H(M; Q_\beta)| = 0 \implies m' = -m|Q_\beta|.$$

This shows that

$$M|_{E(\{Q_\alpha,Q_\gamma\})} \simeq I^n k[-n|Q_\alpha|]|_{E(\{Q_\alpha,Q_\gamma\})} \quad \text{and} \quad M|_{E(\{Q_\beta,Q_\gamma\})} \simeq I^m k[-m|Q_\beta|]|_{E(\{Q_\beta,Q_\gamma\})}$$

Lastly, using the two modules above to compute $H(M; Q_\gamma)$, we see that

$$n(|Q_\gamma| - |Q_\alpha|) = m(|Q_\gamma| - |Q_\beta|).$$

This proves Lemma 7.1. $\square$

Using *Lemma* 7.1, we described all restrictions of M using just one unknown, either n or m. Now, we try the same approach as previously: casework on the annihilator A_x. Let $x \in M$ be an element with minimal degree. We cannot apply Lemma 6.2 because $M|_{E(S)}$ is not equivalent to k for all subsets $S \subset N$. Nevertheless, in the proceeding Lemma 7.2, we characterize it using $M|_{E(\{Q_\alpha,Q_\beta\})} \simeq k|_{E(\{Q_\alpha,Q_\beta\})}$.

Lemma 7.2. *If M is an $E(N)$ module such that $M|_{E(\{Q_\alpha,Q_\beta\})} \simeq k|_{E(\{Q_\alpha,Q_\beta\})}$ and $x \in M$ has minimal degree, then A_x either contains both Q_α and Q_β or neither. That is, either $Q_\alpha x = Q_\beta x = 0$ or $Q_\alpha Q_\beta x \neq 0$.*

The proof for Lemma 7.2 is found in Appendix A.1.

First we examine the case where $Q_\alpha Q_\beta x \neq 0$. There are two further cases: either $A_x = \{\varnothing\}$ or $A_x = \{Q_\gamma\}$. If A_x is empty, Lemma 5.2 implies the existence of a free module F over $E(N)$ such that $M = M' \oplus F$. We pick another minimal element from M' and calculate its annihilator again. If $A_x = \{Q_\gamma\}$, Lemma 7.3 shows that a contradiction arises due to the degree conditions of the generators.

Lemma 7.3. *If M is an $E(N)$ module such that $M|_{E(\{Q_\alpha,Q_\beta\})} \simeq k|_{E(\{Q_\alpha,Q_\beta\})}$, $x \in M$ has minimal degree, then A_x cannot be $\{Q_\gamma\}$.*

The proof for Lemma 7.3 is found in Appendix A.2, which uses an inequality on the degrees of Q_α, Q_β, and Q_γ to deduce the desired result. Therefore, in the case that $Q_\alpha Q_\beta x \neq 0$, we can always reduce M to a submodule M'.

We are left with the case $Q_\alpha x = Q_\beta x = 0$. If $Q_\gamma x = 0$, then $M \simeq k$ by Lemma 5.1 and we are done. Otherwise, $Q_\gamma x \neq 0$. This is the only possibility left to analyse. If Theorem 4.2 is true for all $|N| \geq 3$, then this possibility is impossible. However, it is unknown how to create a contradiction arising from $Q_\gamma x \neq 0$. There may even exist a counterexample to Theorem 4.2 when $|N| = 3$, similar to the Joker module mentioned in Section 1. The investigation of this case will be the subject of future work.

8 Ackowledgements

I would like to extend my deepest gratitude towards my mentor David Lee for suggesting this project and his invaluable guidance througout the past 6 weeks. I would also like to thank Dr. Tanya Khovanova of the Department of Mathematics at the Massachusetts Institute of Technology for her advice on early drafts of the paper. My appreciation also extends to my tutor Peter Gaydarov, who gave me countless advise regarding the presentation of my project; my last week teaching assistant Allen Lin for all his suggestsions; and Dr. John Rickert for his feedback on the mathematical organization of this project. Thank you to the Center for Excellence in Education (CEE), the Massachusetts Institute of Technology (MIT), and the Research Science Institute (RSI) for giving me this amazing opportunity. Finally, I would like to express my gratitude to my sponsors.

References

[1] Salix Alba. "Winding circles". `https://upload.wikimedia.org/wikipedia/commons/3/3d/Fundamental_group_of_the_circle.svg`. CC BY-SA 3.0.

[2] N. Johnson. Hopf Fibration. `https://upload.wikimedia.org/wikipedia/commons/b/b9/Hopf_Fibration.png`. CC BY-SA 3.0.

[3] J. Adams. On the Structure and Applications of the Steenrod Algebra. *Commentarii Mathematici Helvetici*, 32:180–214, Dec. 1958. Received 07 June 1957.

[4] J. Adams and S. Priddy. Uniqueness of BSO. *Mathematical Proceedings of the Cambridge Philosophical Society*, 80(3):475–509, 1976.

[5] P. Bhattacharya and N. Ricka. The Stable Picard Group of $\mathcal{A}(2)$. *arXiv preprint arXiv:1702.01493*, 2017.

[6] J. Pan and R. Yan. The Stable Picard Group of A(n). *arXiv preprint arXiv:2212.09985*, 2022.

[7] J. Milnor. The Steenrod Algebra and Its Dual. *Annals of Mathematics*, 67(1):150–171, 1958.

[8] J. Adams and H. Margolis. Modules over the Steenrod algebra. *Topology*, 10(4):271–282, 1971.

[9] C. A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

A Appendix

A.1 Proof of Lemma 7.2

Proof. Suppose $Q_\alpha x = 0$. If x is also in the image of Q_α, then there exists some element $y \in M$ such that $x = Q_\alpha y$. This element y has degree $|x| - |Q_\alpha|$, which contradicts the minimality of x. Thus, x is not in the image of Q_α and it generates $H(M; Q_\alpha)$. Because $M|_{E(\{Q_\alpha, Q_\beta\})} \simeq k|_{E(\{Q_\alpha, Q_\beta\})}$, the Margolis homology $H(M; Q_\alpha)$ is k in degree 0 by Lemma 2.3. This implies that $|x| = 0$. Furthermore, Lemma 5.1 shows that there must exist a generator x' of both $H(M; Q_\alpha)$ and $H(M; Q_\beta)$ in degree 0. This x' is a scalar multiple of x. Therefore $Q_\beta x = 0$. Thus, either $Q_\alpha x = Q_\beta x = 0$ or $Q_\alpha Q_\beta x \neq 0$. $\square$

Suppose A_x does not contain either Q_α or Q_β.

A.2 Proof of Lemma 7.3

Proof. Suppose M is an $E(N)$ module such that $M|_{E(\{Q_\alpha, Q_\beta\})} \simeq k|_{E(\{Q_\alpha, Q_\beta\})}$ and $x \in M$ has minimal degree. Note that

$$Q_\gamma(Q_\beta Q_\alpha)x = (Q_\beta Q_\alpha)Q_\gamma x = 0 \implies Q_\beta Q_\alpha x \in \ker Q_\gamma.$$

If $Q_\beta Q_\alpha x$ is also in the image of Q_γ, then there exists another element y such that $Q_\beta Q_\alpha x = Q_\gamma y$. However, Theorem 6.3 implies that y has a degree lower than x, which is impossible. Thus, $Q_\beta Q_\alpha x$ generates the Margolis homology $H(M; Q_\gamma)$. Using Lemma 2.3, we calculate $H(M; Q_\gamma)$ to be k in degree $n(|Q_\gamma| - |Q_\alpha|) = m(|Q_\gamma| - |Q_\beta|)$. Any generator of the homology must also be in this degree. Therefore,

$$\begin{aligned} |Q_\beta Q_\alpha x| = n(|Q_\gamma| - |Q_\alpha|) \implies |x| &= n(|Q_\gamma| - |Q_\alpha|) - |Q_\beta Q_\alpha| \\ &= n(|Q_\gamma|) - (n+1)|Q_\alpha| - |Q_\beta| \\ &= n(2p^\gamma - 1) - (n+1)(2p^\alpha - 1) - 2p^\beta + 1 \\ &= 2(n(p^\gamma - p^\alpha) - p^\alpha - p^\beta + 1) \neq 0. \end{aligned}$$

Note that this final expression is positive when n is positive, and negative when n is negative. However, this contradicts the fact that the element of minimum degree in $M|_{E(\{Q_\beta, Q_\gamma\})} \simeq I^m k[-m|Q_\beta|]|_{E(\{Q_\beta, Q_\gamma\})}$ has positive degree when n is negative, and has negative degree when n is positive. Therefore, it is impossible that $A_x = \{Q_\gamma\}$. $\square$